Notes on

A-Level Physics

Ma Weiyi

ma.weiyi@dhs.edu.sg

These notes review the H2 Physics (9749) and H3 Physics (9814) syllabus.

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Contents

Part I

Measurement

1 Measurements

1.1 Physical Quantities

Definition 1.1: Quantity

A **physical quantity** is any scientifically measurable quantity (e.g. temperature, force, mass). All physical quantities consist of a **numerical value** and a **unit**.

A **base quantity** is one of the seven physical quantities of the S.I. system by which all other physical quantities are defined.

A **derived quantity** is a physical quantity that can be expressed in terms of products and quotients of base quantities.

The seven base quantities, along with their S.I. units, are shown below.

Base quantity	Base unit	Symbol
Mass	kilogram	kg
Length	metre	m
Time	second	s
Electric current	ampere	A
Thermodynamic temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

Base units can be used to **check the dimensional homogeneity** of a physical equation. If the units of each term are equal on both sides of the equation, the equation is said to be **homogenous** or dimensionally constant.

Quantities use prefix to indicate decimal sub-multiples or multiples.

1.2 Scalars & Vectors

Definition 1.2: Scalars & Vectors

A **scalar** is a quantity has **magnitude** but **no direction**. It is completely specificed by its numerical value and unit.

A **vector** is a quantity that has **both** magnitude and direction. It must be specified with its value, unit, and direction.

The **change** in a certain physical quantity $\vec{\mathbf{Q}}$ can simply be taken as the subtraction between its final and initial state.

$$\Delta\mathbf{Q} = \mathbf{Q}_f - \mathbf{Q}_i$$

10^n	Prefix	Symbol	Name
10^12	tera	T	trillion
10^9	giga	G	billion
10^6	mega	M	million
10^3	kilo	k	thousand
10^0	-	-	one
10^{-1}	deci	d	tenth
10^{-2}	centi	c	hundredth
10^{-3}	milli	m	thousandth
10^{-6}	micro	μ	millionth
10^{-9}	nano	n	billionth
10^{-12}	pico	p	trillionth

Since two vectors can be added to give a resultant vector, any vector can be resolved into two constituent vectors.

Vectors can be resolved into **two mutually perpendicular, independent components** through trigonometry. Given a vector $\vec{\mathbf{A}}$ with magnitude $|\vec{\mathbf{A}}|$ inclined at an angle θ from the positive x-axis, its corresponding x and y components are given as:

$$A_x = |\mathbf{A}|\cos\theta, \qquad A_y = |\mathbf{A}|\sin\theta$$

1.3 Errors & Uncertainties

Definition 1.3: Uncertainty

Uncertainty is the range of values of a measurement in which the actual value of the measurement is expected to lie within, commonly arising from (1) limitations of the observer, (2) limitations of the apparatus, and/or (3) limitations of the method.

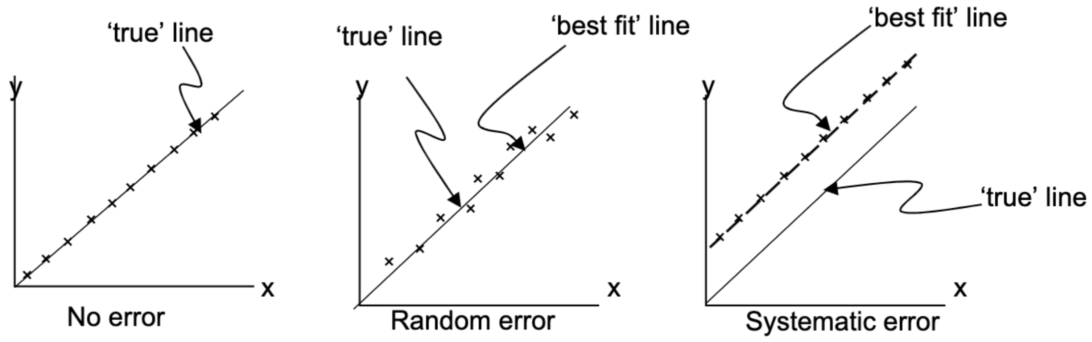
Definition 1.4: Error

Error is the difference between the measured value and the true value.

Systematic errors are present when the measured values produce errors of the same magnitude and sign. They cannot be eliminated by averaging, but can be reduced through careful design of experimental processes.

Random errors are present when the measured values produce errors of different magnitudes and signs. These readings are scattered about the **mean value** with no fixed pattern.

As described, errors can be differentiated into systematic errors and random errors.



1.3.1 Precision & Accuracy

Definition 1.5: Precision

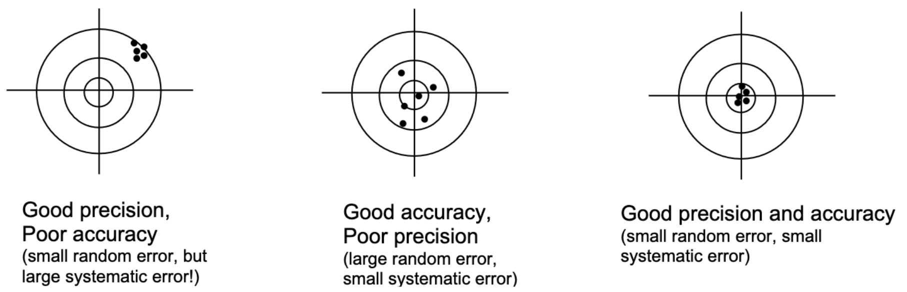
The **precision** of a measurement refers to how close the experimental values are to each other.

Precision describes the level of uncertainty in an instrument's scale. Good precision indicates small random errors.

Definition 1.6: Accuracy

Accuracy is the closeness of a reading on an instrument to the true value of the quantity being measured.

Good accuracy indicates small systematic errors.



1.3.2 Uncertainty

In a scale reading, the **absolute uncertainty** is half of its smallest division, rounded off to 1 significant figure. In general, all readings can be recorded in the form $R \pm \Delta R$, where ΔR is the absolute uncertainty.

The uncertainty of a measured value can also be presented as a percent or a simple fraction. The **fractional** uncertainty of R is $(\Delta R)/R$, whereas its **percentage** uncertainty is $(\Delta R)/R \times 100\%$.

The absolute uncertainty (with units, 1 s.f.) indicates the **scale sensitivity** of the measuring instrument used. The percentage uncertainty (dimensionless, 2 s.f.) is useful to compare the **significance of the error**.

1.4 Propagation of Uncertainties

Consequential uncertainty is the overall estimate of uncertainty throughout an entire experiment. To statistically calculate consequential uncertainty, we consider **maximum uncertainty**, which occurs in the worst-case scenario.

Let A and B be measured independent quantities, with ΔA and ΔB being their corresponding uncertainties. The following rules will apply for a derived quantity Z.

$$Z = mA + nB \implies \Delta Z = |m|\Delta A + |n|\Delta B$$

$$Z = A^m \times B^n \implies \frac{\Delta Z}{Z} = |m|\frac{\Delta A}{A} + |n|\frac{\Delta B}{B}$$

When adding or substracting measurements, add their absolute uncertanities. When multiplying or dividing measurements, add their fractional uncertainties. If a quantity A is being divided by k, simply regard it as a multiplication of k^{-1}.

For complicated functions, it is acceptable to calculate the maximum uncertainty as the median value between the range of possible quantities.

$$\Delta Z = \frac{1}{2}(Z_{\max} - Z_{\min})$$

Part II

Newtonian Mechanics

2 Kinematics

2.1 Translational Kinematic Quantities

Definition 2.1: Distance

Distance, d, is the total length of path covered by a moving object irrespective of the direction of motion. It is a scalar quantity.

Definition 2.2: Displacement

Displacement, denoted by $\vec{\mathbf{s}}$, is the linear distance of the position of a moving object from a given reference point. It is a vector quantity.

Definition 2.3: Speed

Speed is the distance travelled per unit time. It is a scalar quantity.

The **average speed**, $\langle u \rangle$, is the ratio of total distance travelled to the time interval during which a particle travels over that distance.

$$\langle u \rangle = \frac{d}{t}$$

Definition 2.4: Velocity

Velocity, denoted by $\vec{\mathbf{v}}$, is the rate of change of displacement with time, acting in the direction of change of displacement. It is a vector quantity.

The **average velocity**, $\langle v \rangle$, is the ratio of displacement to the time interval during which the displacement occurs.

$$\langle v \rangle = \frac{s}{t}$$

Instantaneous velocity is defined as the limit of the average velocity as the time interval Δt becomes infinitesimally short. This is mathematically formulated below.

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

Definition 2.5: Acceleration

Acceleration, denoted by $\vec{\mathbf{a}}$, is the rate of change of velocity with time.

Acceleration is a *vector quantity* that acts in the direction of the change of velocity. An object is said to be accelerating if its velocity is changing, either in magnitude or direction.

Average acceleration is the ratio of the change in velocity over the time interval for which this change occurs.

$$\langle a \rangle = \frac{\Delta v}{\Delta t}$$

Instantaneous acceleration is the limit of average acceleration as the time interval goes to zero.

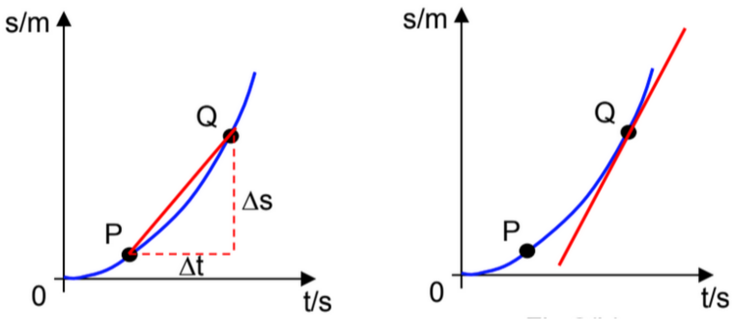
$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

2.2 Graphical Methods

2.2.1 Displacement-Time Graphs

The displacement of any object can be represented using a displacement-time graph, which expresses how displacement changes with time.

Since $v = \dfrac{ds}{dt}$, the velocity at any instant can be found from the gradient of the displacement-time graph.



The ratio $\dfrac{\Delta s}{\Delta t}$ shown on the left graph gives the magnitude of the **average velocity**, $\langle v \rangle$, for the time interval Δt.

The instantaneous velocity at Q, shown on the right graph, is found by finding the gradient of the tangent at point Q.

2.2.2 Velocity-Time Graphs

We can also represent the motion of an object using a velocity-time graph.

If two bodies have positions at times (x_1, t_1) and (x_2, t_2) respectively, the displacement Δx during a small time interval Δt is equal to $\langle v \rangle_x \Delta t$, where $\langle v \rangle_x$ is the average x-velocity during Δt. The total displacement $x = x_2 - x_1$ during the interval $[t_1, t_2]$ is thus given by

$$x_2 - x_1 = \int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v_x \, dt$$

The displacement is the **time integral** of x-velocity V_x. Graphically, the displacement between a time interval is the **area under the v_x-t curve** within that time interval.

Since $a = \dfrac{dv}{dt}$, the acceleration at any instant can be found from the gradient of the velocity-time graph.

2.3 Equations of Motion

Consider a particle moving with constant acceleration, a, in a straight line. Let u be the initial velocity, v be the final velocity, t be the time taken, and s be the displacement along a straight line.

The **kinematic equations of motions** in such a situation are

$$\begin{aligned} v &= u + at \\ s &= 0.5(u+v)t \\ s &= ut + 0.5at^2 \\ s &= vt - 0.5at^2 \\ v^2 &= u^2 + 2as \end{aligned}$$

The equations of motion are vector equations, thus possessing magnitude and direction. For all rectilinear vector problems, directions are indicated using positive and negative signs.

If we take the direction towards our right as positive, then the direction towards our left will be negative.

2.4 Motion Under Gravity

2.4.1 Negligible Air Resistance

Consider scenarios where the only force acting on the ball is gravity. After time t, the velocity of the ball released from rest would be given by

$$\begin{aligned} v &= u + at \\ &= gt \quad (\text{since } u = 0) \end{aligned}$$

2.4.2 Air Resistance

Air resistance opposing motion tends to increase approximately with the square of the speed at high velocities and increase proportionately with speed at low velocities.

The larger the velocity, the greater the air resistance. The actual acceleration is the combined free-fall acceleration and the retardation due to air resistance.

Taking downward as positive, where F_R represents air resistance,

$$\begin{aligned} F = ma &= mg - F_R \\ \therefore a &= g - \frac{F_R}{m} \end{aligned}$$

For bodies falling from great heights, the air resistance can be very large, to the point where $F_R = mg$. Here, $F = 0N$, and the body has reached **terminal velocity**, v_T.

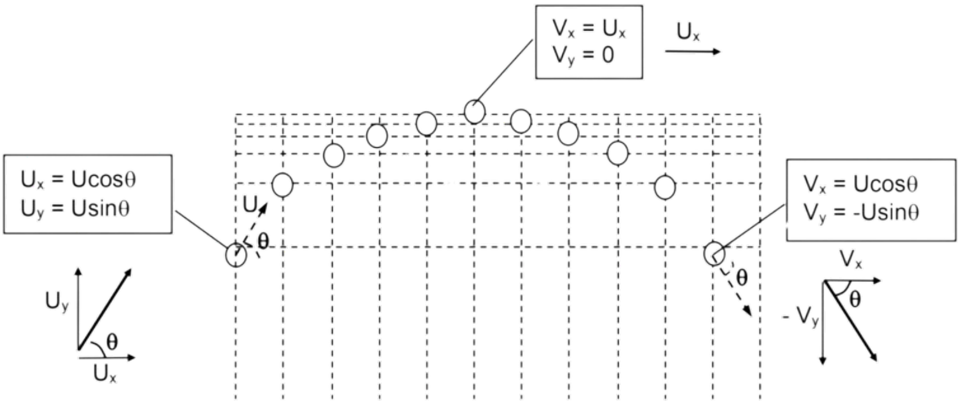
2.5 Non-Linear Motion

Non-linear motion deals with objects moving in a two-dimensional plane, as opposed to straight-line paths. Bodies undergoing projectile motion experience uniform velocity in one direction and a uniform acceleration in a perpendicular direction.

When an object is thrown into the air at an angle θ with respect to the ground with speed v, the body will move in a parabolic path.

The velocity of the projectile at any time is given by the vector sum of v_x and v_y at that time.

$$v = \sqrt{{v_x}^2 + {v_y}^2}$$



To ascertain the angle of direction,

$$\theta = \tan^{-1} \frac{v_y}{v_x}$$

2.5.1 Horizontal Motion

Horizontally, the body is moving with a **constant velocity**. There is no resultant force, hence the horizontal acceleration is **zero**.

$$\begin{aligned} a_x &= 0 \\ v_x &= u_x + a_x t = u_x \\ s_x &= u_x t + 0.5 a_x t^2 = u_x t \end{aligned}$$

2.5.2 Vertical Motion

Vertically, the body moves under the influence of gravity with a **constant acceleration** (of g), causing the velocity to change with time.

Taking downwards as positive,

$$\begin{aligned} a_y &= g \\ v_y &= u_y + gt \\ s_y &= u_y t + 0.5 g t^2 \end{aligned}$$

The following equation is time-independent.

$${v_y}^2 = {u_y}^2 + 2 g s_y$$

2.5.3 Maximum Height

Note that at the maximum height, $v_y = 0$. Taking the time-independent equation,

$$\begin{aligned} {v_y}^2 = {u_y}^2 + 2 g s_y &\implies {v_y}^2 (u \sin\theta)^2 - 2gH \\ &\implies \boxed{H = \frac{u^2 \sin^2\theta}{2g}} \end{aligned}$$

To determine the time taking to reach the maximum height,

$$\begin{aligned} v_y = u_y + a_y t &\implies u \sin\theta - g t_{\text{up}} \\ &\implies \boxed{t_{\text{up}} = \frac{u \sin\theta}{g}} \end{aligned}$$

2.5.4 Total Time of Flight

As the trajectory is a parabola, the time taken to travel upwards is equal to the time taken to travel downwards. Therefore,

$$\begin{aligned} t_{up} = t_{down} &\implies T = 2t_{\text{up}} \\ &\implies \boxed{T = \frac{2u\sin\theta}{g}} \end{aligned}$$

2.5.5 Range

The maximum horizontal distance covered by the projectile is known as the **horizontal range**, R.

$$R = v_x T = (u\cos\theta)\frac{2u\sin\theta}{g}$$

$$\implies \boxed{R = \frac{u^2\sin 2\theta}{g}}$$

For a given speed of projection, u, the horizontal range is maximum when $\sin 2\theta = 1 \Rightarrow \theta = 45°$.

Thus, the maximum range is

$$R_{\text{max}} = \frac{u^2}{g}$$

3 Dynamics

3.1 Mass, Weight, & Momentum

Definition 3.1: Mass

Mass is the amount of matter in a body. It is a scalar quantity with S.I. unit kg.

Consequently, mass is the measure of inertia in the body — the **inertia** of a body is the reluctance of the body to start moving, and to stop once it has begun moving.

The larger the mass of a body, the larger its inertia.

Definition 3.2: Weight

Weight is defined as the **force** acting on a body due to **gravity**. For any given non-massless body,

$$\vec{\mathbf{W}} = m\vec{\mathbf{g}}$$

where $\vec{\mathbf{W}}$ is the weight of the body, m is the mass of the body, and $\vec{\mathbf{g}}$ is the acceleration due to gravity.

The mass of an object is constant irrespective of space, but its weight is a **force** with a magnitude dependent on the value of g.

An object's weight on the Moon is a fraction of that on Earth, due to the Moon's weaker force of gravity. Nonetheless, it retains its inertia, as the mass is the same.

Definition 3.3: Linear Momentum

Linear momentum of a body is the product of its mass and velocity.

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$$

where $\vec{\mathbf{p}}$ is the momentum, m is the mass, and $\vec{\mathbf{v}}$ is the velocity of the body.

Momentum is a vector quantity acting in the same direction as velocity. The S.I. unit of momentum is N s.

3.2 Newton's Laws of Motion

Definition 3.4: Newton's First Law

Newton's first law of motion states that a body continues in its state of rest or uniform motion in a straight line, unless a resultant external force acts on it.

This law implies that

1. the **state of rest** requires no resultant force to maintain;
2. the **state of uniform velocity** also requires no resultant force to maintain.

Definition 3.5: Newton's Second Law

Newton's second law of motion states that the rate of change of momentum of a body is proportional to the resultant force acting on it, and occurs in the direction of the resultant force.

From the above definition, we see that

$$\sum \vec{\mathbf{F}} \propto \frac{d\vec{\mathbf{p}}}{dt} \implies \sum \vec{\mathbf{F}} = k\frac{d\vec{\mathbf{p}}}{dt}$$

where k is the constant of proportionality.

The S.I. unit force is the newton (N). One newton is defined as the force which produces an acceleration of 1m/s^2 when applied to a mass of 1kg, thus conveniently setting $k = 1$.

$$\begin{aligned}\sum \vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} &= \frac{d(m\vec{\mathbf{v}})}{dt} \\ &= m\frac{d\vec{\mathbf{v}}}{dt} + \vec{\mathbf{v}}\frac{dm}{dt}\end{aligned}$$

When **mass** is constant,

$$\sum \vec{\mathbf{F}} = m\frac{d\vec{\mathbf{v}}}{dt} = m\vec{\mathbf{a}}$$

When **velocity** is constant,

$$\sum \vec{\mathbf{F}} = \vec{\mathbf{v}}\frac{dm}{dt}$$

A direct result of Newton's Second Law is the **impulse-momentum theorem**. Here, we define the impulse of a force and formulate the impulse-momentum theorem.

Definition 3.6: Impulse

The **impulse of a force** is defined as the time integral of a force during which the force is acting on an object.

$$\vec{\mathbf{I}} = \int_{t_i}^{t_f} \sum \vec{\mathbf{F}}\, dt$$

Definition 3.7: Impulse-Momentum Theorem

The **impulse-momentum theorem** states that the impulse of a force acting on an object is equal to the **change** in the momentum of the object.

$$\int_{t_i}^{t_f} \sum \vec{\mathbf{F}}\, dt = \Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i$$

Proof. From Newton's Second Law,

$$\begin{aligned}\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} &\implies \vec{\mathbf{F}}dt = d\vec{\mathbf{p}} \\ &\implies \int_{t_i}^{t_f} \vec{\mathbf{F}}\, dt = \int_{\vec{\mathbf{p}}_i}^{\vec{\mathbf{p}}_f} d\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i = \Delta\vec{\mathbf{p}}\end{aligned}$$

□

The extent to which the momentum of a body is changed by a given net force depends on the magnitude of the net force, and the duration for which the force acts on the body.

Definition 3.8: Newton's Third Law

Newton's third law of motion states that if a body A exerts a force on a body B, then body B exerts an equal but opposite force on body A.

$$F_{AB} = -F_{BA}$$

where F_{AB} is the force that A exerts on B, and F_{BA} is the force that B exerts on A.

3.3 Common Mechanical Forces

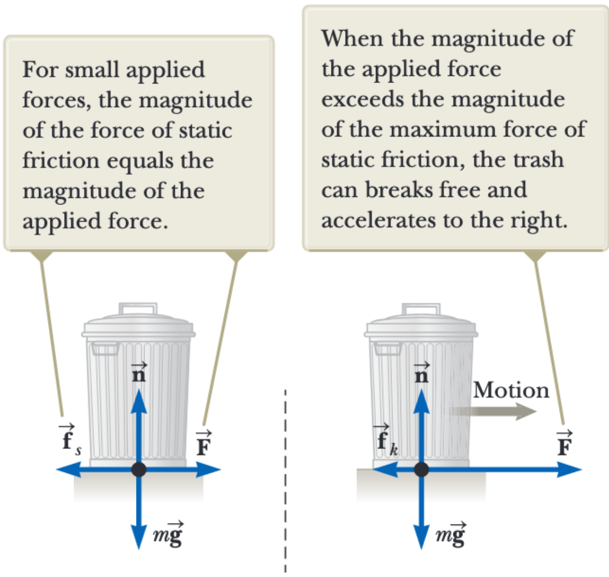
We discuss mechanical forces that are commonly considered when drawing free-body diagrams.

3.3.1 Normal Force

The **normal force** is the **contact** force that acts perpendicular to the surface between two objects. When we are standing on the ground, the normal force on us due to the ground opposes the gravitational force and prevents us from accelerating to the Earth's core.

3.3.2 Friction

Between any two surfaces in contact, there will always be a normal force (perpendicular to the interface) and a frictional force (parallel to the interface).



Static friction, $\vec{f}_s$, is always directed to oppose the **relative impending motion**.

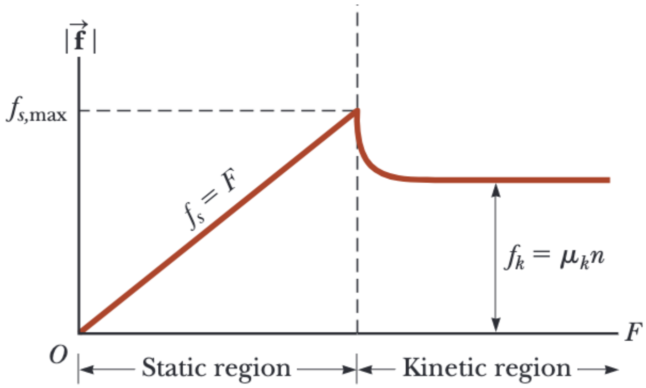
$$f_s \leq \mu_s n$$

where μ_s is the coefficient of the static frictional force.

Kinetic friction, $\vec{f}_k$, is always directed to oppose the **relative motion**.

$$f_k = \mu_k n$$

where μ_k is the coefficient of the kinetic frictional force.



Values of μ_s and μ_k depend on the nature of the surfaces, but μ_k is generally less than μ_s. μ_s and μ_k are nearly independent of the area of contact.

3.3.3 Tension

Tension is the force exerted by a rope when pulled on. Every piece of the rope feels a tension force in both directions, except for the endpoints, which feel a tension on one side and a separate force by the object on the other.

Note that in most cases, ropes are assumed to be **massless**, hence the **tension must be the same at all points**. Otherwise, this would yield a net force, consequently resulting in infinite acceleration.

3.4 Free-Body Diagrams

When considering problems, it is helpful to draw a free-body diagram with the following procedure:

1. Isolate the system in question.
2. Identify all external forces acting on the system and draw them as vectors as appropriate points on the system.

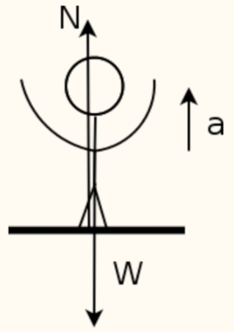
Then, follow the general procedure for solving.

1. Consider **various systems** and their free-body diagrams. To choose systems, examine the forces that you need to solve. If they do not include internal forces (e.g. friction, normal forces between surfaces), consider **paired objects** as a **whole system**. Otherwise, segregate the objects.
2. Write the $\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$ equations for all considered systems.
3. If necessary, identify certain relationships between variables to formulate additional equations.

Example 3.1

A man of mass m is in a lift undergoing an acceleration a upwards. Find the normal force exerted on the man by the floor of the lift.

We start by drawing a free-body diagram.

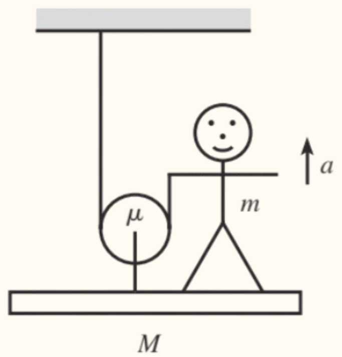


From Newton's Second Law, we obtain

$$N - mg = ma \implies \boxed{N = m(a+g)}$$

Example 3.2

A person stands on a platform-and-pulley system known as a **Bosun's chair**. The masses of the platform, person, and pulley are M, m, and μ respectively. The rope is massless. Let the person pull up on the rope so that she has acceleration a upward. Find the tension of the rope, the normal force between the person and the platform, and the tension in the rod connecting the pulley to the platform.



To find the tension in the rope, consider the whole system (except the ceiling). The only forces acting on this system are the three weights (Mg, mg and μg) and the tension T.

$$T - (M + m + \mu)g = (M + m + \mu)a \implies \boxed{T = (M + m + \mu)(g + a)}$$

To find the normal force N between the person and he platform, and also the tension f in the rod connecting the pulley to the platform, we must consider subsystems.

Applying Newton's 2nd law to the person (experiencing gravity, normal force, and tension),

$$N - T - mg = ma$$

Applying Newton's 2nd law to the platform (gravity, normal force, and force upward from the rod),

$$f - N - Mg = Ma$$

Applying Newton's 2nd law to the pulley (gravity, force downward from the rod, and twice the tension in the rope),

$$2T - f - \mu g = \mu a$$

From these equations, we can deduce

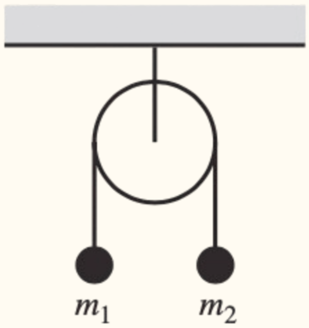
$$\boxed{N = (M + 2m + \mu)(g + a)}$$

and

$$\boxed{f = (2M + 2m + \mu)(g + a)}$$

Example 3.3

Consider the pulley system below, termed an **Atwood's machine**, with masses m_1 and m_2. The strings and pulleys are massless. Solve for the accelerations of the masses and the tensions of the strings.



Firstly, since the strings are massless, tension must be constant throughout (otherwise infinite acceleration occurs).

$$T_1 = T_2$$

Applying Newton's 2nd law,

$$T - m_1 g = m_1 a$$
$$T - m_2 g = m_2 a$$

The final equation relating a_1 to a_2 is the "conservation of string". If we move the m_2 pulley upwards by a distance d, the m_1 pulley has to go down a distance d.

This applies to acceleration as well.

$$a_1 = -a_2$$

Thus, this gives

$$T = \frac{2m_1 m_2}{m_1 + m_2} g$$

$$a_1 = \frac{m_1 - m_2}{m_1 + m_2} g$$

$$a_2 = \frac{m_2 - m_1}{m_1 + m_2} g$$

3.5 Conservation of Linear Momentum

Definition 3.9: Principle of Conservation of Linear Momentum

The **principle of conservation of linear momentum** states that the total momentum of a system **remains constant**, provided that **no** net external resultant force acts on the system.

Consider a mass m_A travelling with velocity u_A that collides with mass m_B with velocity u_B. During the collision, the two masses are in contact and exert a force F on each other. The duration of impact is Δt.

By Newton's 2nd Law,

$$F_{AB} = \frac{\Delta p_B}{\Delta t} = \frac{m_B v_B - m_B u_B}{\Delta t}$$

$$F_{BA} = \frac{\Delta p_A}{\Delta t} = \frac{m_A v_A - m_A u_A}{\Delta t}$$

By Newton's 3rd Law,

$$F_{AB} = -F_{BA}$$

$$\frac{m_B v_B - m_B u_B}{\Delta t} = -\frac{m_A v_A - m_A u_A}{\Delta t}$$

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B$$

$$\therefore \text{total initial momentum} = \text{total final momentum}$$

With this principle in mind, we may examine two main types of collisions between bodies — **elastic collisions** and **inelastic collisions**.

In elastic collisions, the **total momentum and kinetic energy is conserved**. As such, the relative velocity of approach is equal to the relative velocity of separation.

In inelastic collisions, only the total momentum is conserved. The **total kinetic energy is not conserved**.

3.5.1 Elastic Collisions

Consider two spheres, A and B with respective masses m_A and m_B, **colliding elastically** with each other. The initial velocities of A and B are u_A and u_B respectively. After the collision, A and B move off with velocities v_A and v_B respectively.

By the conservation of momentum,

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B$$

By the conservation of kinetic energy,

$$\frac{1}{2} m_A {u_A}^2 + \frac{1}{2} m_B {u_B}^2 = \frac{1}{2} m_A {v_A}^2 + \frac{1}{2} m_B {v_B}^2$$

Following from this,

$$\begin{aligned} m_A(u_A - v_A) &= m_B(v_B - u_B) && \text{(from COM)} \\ m_A(u_A + v_A)(u_A - v_A) &= m_B(v_B + u_B)(v_B - u_B) && \text{(from COKE)} \\ \implies u_A + v_A &= v_B + u_B \\ \therefore u_A - u_B &= v_B - v_A \end{aligned}$$

This shows us that for elastic collisions, the **relative velocity of approach is equal to the relative velocity of separation** in terms of magnitude. Consequently, we may derive the closed-form expressions for the final velocities for each body.

$$v_1 = \frac{m_A - m_B}{m_A + m_B} u_A + \frac{2m_B}{m_A + m_B} u_B$$

$$v_2 = \frac{2m_A}{m_A + m_B} u_A + \frac{m_B - m_A}{m_A + m_B} u_B$$

From these results, we may highlight expected behaviour in special, notable situations.

If **two identical masses** collide, they exchange velocities after the collision.

$$v_1 = u_2 \quad \text{and} \quad v_2 = u_1$$

If an **extremely big** mass, m_1, collides with a **very small mass at rest**, m_2 (i.e. $m_1 >> m_2$ and $v_2 = 0$), the big mass continues with a speed close to its initial speed, while the small mass bounces off with a speed about twice the initial speed of m_1.

$$v_1 \approx u_1 \quad \text{and} \quad v_2 \approx 2u_1$$

If a **very small** mass, m_1, collides with a **very big mass at rest** (i.e. $m_2 >> m_1$ and $v_2 = 0$), the small mass rebounds with a speed close to its initial speed, and the big mass remains almost stationary.

$$v_1 \approx -u_1 \quad \text{and} \quad v_2 \approx 0$$

3.5.2 Inelastic Collisions

In realistic collisions, change in kinetic energy take place although the momentum of the system is still conserved. This loss in kinetic energy is due to the increase in internal energy and heat dissipated to the surroundings.

For **inelastic collisions**,

1. total momentum is conserved;

$$m_A u_A + m_B u_B = m_A v_A + m_B v_B$$

2. and total kinetic energy is not conserved.

$$\frac{1}{2} m_A {u_A}^2 + \frac{1}{2} m_B {u_B}^2 \neq \frac{1}{2} m_A {v_A}^2 + \frac{1}{2} m_B {v_B}^2$$

For **perfectly inelastic collisions**, the two bodies will coalesce post-collision, moving off together with a common velocity.

$$v_1 = v_2$$

4 Forces

4.1 Conservative Forces

A **force** is defined as the rate of change of momentum of a free-moving object. The direction of the force is in the direction of the change of momentum.

A force is **conservative** if the work it does on an object moving between two points is **independent** of the path the object takes.

Conversely, a force is **non-conservative** if the work it does on an object depends on the path taken by the object between its endpoints.

For instance, gravity is a conservative force, while friction is a non-conservative force.

4.2 Dissipative Forces

Friction and viscous forces are known as **dissipative forces**.

Some energy of the object moving through the fluid is **dissipated as heat** to the surroundings.

4.2.1 Friction

Definition 4.1: Frictional Force

Frictional force is the force exerted by one body on another body when two bodies slide over one another.

It is caused by irregularities in the surfaces in mutual contact and depends on the surfaces in contact as well as how much they are pressed against each other. It is the **component of contact force** along surface of contact.

Even the flattest and most highly polished surface have hollows and humps more than one hundred atoms high — when one solid is placed over one another, contact occurs only at a few places of small areas. The pressure at the points of contact is extremely high and causes the humps to flatten out until the increased area of contact enables the upper solid to be supported.

At the points of contact, small, cold-welded "joints" are formed by the strong adhesive forces between molecules which are very close together. These joints have to be broken before one surface can move over the other, giving way to **static friction**.

4.2.2 Viscous Drag

Definition 4.2: Drag

Viscous drag is the frictional force experienced either by an object as it moves through a fluid or by a fluid as it moves over a surface.

The magnitude of drag **increases** as the speed of the object increases.

4.3 Center of Gravity

Every particle of mass is attracted towards the centre of the earth by force of gravity.

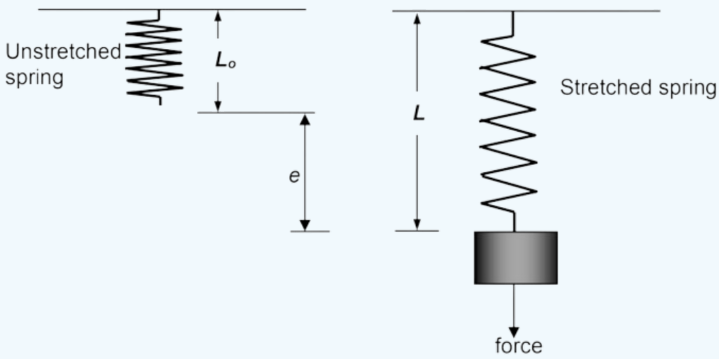
The weight of a body may be taken **as acting at a single point**, known as its centre of gravity (hereafter termed C.G.).

The C.G. of a homogeneous, symmetric body must lie on its axis of symmetry.

4.4 Hooke's Law

Definition 4.3: Hooke's Law

Hooke's law states that the change in length e of a material is directly proportional to the resultant force F applied, provided the limit of proportionality has not been exceeded. For the given diagram,



$$F = kx = k(L - L_0)$$

where

- F is the force applied to the material
- L_0 is the unstretched length of material
- L is the final length of material
- x is the extension of the material, where $e = L - L_0$
- k is the proportionality constant

Assuming that a material obeys Hooke's law, the force applied is **linearly proportional** to the extension of the material. This law applies to springs, as well as metals in the form of wires.

For a number of spring constants $k_1, k_2, \cdots, k_n$ that are connected in parallel, they can be replaced by a single spring constant.

$$k_{\text{eff}} = k_1 + k_2 + \cdots + k_n = \sum_{i=1}^{n} k_i$$

For a number of spring constants $k_1, k_2, \cdots, k_n$ that are connected in parallel, they can be replaced by a single spring of constant

$$k_{\text{eff}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} = \left(\sum_{i=1}^{n} \frac{1}{k_i}\right)^{-1}$$

4.4.1 Elastic Potential Energy

Definition 4.4: Elastic Potential Energy

The **elastic potential energy** stored in a deformed material obeying Hooke's law can be found from the **area under a force-extension graph**.

$$U = \frac{1}{2}Fx = \frac{1}{2}kx^2 = \frac{F^2}{2k}$$

An external force F_{ext} acting on a wire causing it to extend from x_1 to x_2 performs work given by

$$W = \int_{x_1}^{x_2} F_{\text{ext}}\, dx$$

This work is stored as **elastic potential energy** in the wire. The elastic potential energy is equal to the area under the force-extension curve, between the limits of x_1 and x_2.

4.5 Upthrust

Definition 4.5: Upthrust

Upthrust is the **net upward force** exerted by the surrounding fluid when a body is **submerged fully or partially** in a fluid.

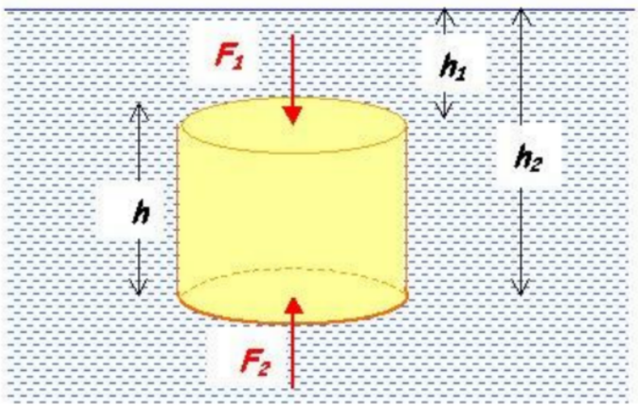
It is the resultant force due to the difference in pressure exerted by the fluid at the top and bottom surfaces of the body. It is equal in magnitude to the weight of fluid displaced by the body.

Definition 4.6: Archimedes's Principle

Archimedes's Principle states that for any object immersed partially or fully in a fluid, the upthrust is equal in magnitude and opposite in direction to the weight of fluid displaced by the object.

$$U = W_{\text{fluid displaced}} = m_f g = \rho_f g V_{\text{fluid displaced}}$$

Proof. Consider a solid cylinder of height h and cross-sectional area A, submerged in a fluid of density ρ.



Recall that pressure is force over area.

$$p = \frac{F}{A}$$

Hence, the pressure of the top surface is given by

$$\begin{aligned} p_{\text{top}} &= \frac{\text{weight of fluid}}{\text{cross-sectional area}} + p_{\text{atm}} \\ &= \frac{m_{\text{top}} g}{A} + p_{\text{atm}} \\ &= \frac{\rho_f A h_1 g}{A} + p_{\text{atm}} \\ &= \rho_f h_1 g + p_{\text{atm}} \end{aligned}$$

Similarly, the pressure on the bottom surface is given by

$$p_{\text{bottom}} = \rho_f h_2 g + p_{\text{atm}}$$

Hence, the **upthrust** (resultant upward force) on the cylinder due to pressure difference between the top and bottom surfaces is

$$\begin{aligned} U &= F_2 - F_1 \\ &= \rho_f g (h_2 - h_1) A \\ &= \rho_f g h A \\ &= \rho_f g V_{\text{fluid displaced}} \end{aligned}$$

$$\therefore U = m_f g$$

This gives us Archimedes's principle. □

Definition 4.7: Principle of Floatation

The **principle of floatation** states that when an object is **floating in equilibrium** in a fluid, the **weight** of the object is equal to the weight of **fluid displaced** by the object.

$$W_{\text{object}} = W_{\text{fluid displaced}}$$

An object floats because the **upthrust** acting on it is equal in magnitude and opposite in direction to the weight of the object. The object sinks when the upthrust acting on it is less than its weight.

A ship made of steel can float because its internal hollow volume displaces a large amount of water, producing sufficient upthrust to keep the ship floating.

4.6 Static Equilibrium

Definition 4.8: Equilibrium

When a body is in **equilibrium**, there is **no resultant force** acting on the body in any direction and no resultant moment acting on the body about any point.

Paraphrasing, the conditions for any body in equilibrium are that

1. the resultant force on it must be zero in any direction;
2. and the resultant torque on it must be zero about any axis of rotation.

4.6.1 Translational Equilibrium

When the resultant force on a body is zero in any direction, there is no acceleration of its centre of mass and the body is said to be in **translational equilibrium**.

If the forces can be resolved into components in two chosen perpendicular directions (such as the x and y axes), then

$$\sum F_x = 0 \text{ and } \sum F_y = 0$$

Similarly, since the resultant force acting on a body is zero, the **vector sum** of forces acting on the body must be zero in any direction as well. The vector diagram showing the addition of all forces acting on the body will be a closed polygon.

4.6.2 Rotational Equilibrium

When the resultant torque of a body is zero in any axis of rotation, there is no angular acceleration of the object and the body is said to be in **rotational equilibrium**. We introduce the notion of moments of a force, as well as the torque of a couple to formalise rotational equilibrium.

Definition 4.9: Moment of a Force

The **moment of a force** about a point is the product of the magnitude of the force and the perpendicular distance of the force from that point.

For a force $\vec{\mathbf{F}}$ acting on a body a perpendicular distance $r_\perp$ away from the pivot, the magnitude of the moment τ is given by

$$\vec{\tau} = \vec{\mathbf{F}} \times \vec{\mathbf{r}} = Fr_\perp$$

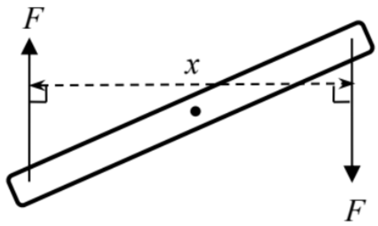
Definition 4.10: Couple

A **couple** consists of a pair of forces of **equual magnitude** but acting in **opposite directions** whose lines of action are **parallel but separate**.

Definition 4.11: Torque of a Couple

The **torque of a couple** is the **product** of one of the forces and the perpendicular distance between the forces.

Considering the diagram below,



The magnitude of the torque of the couple, τ, is given by

$$\tau = \frac{1}{2}Fx + \frac{1}{2}Fx = Fx$$

Definition 4.12: Principle of Moments

The **principle of moments** states that when a system is in equilibrium, the sum of clockwise moments about any axis must be equal to the sum of anticlockwise moments about the same axis.

When the resultant torque of a body is zero about any axis of rotation, it further implies that the **sum of moments of all forces** acting on the body would be zero about any axis.

Hence, the principle of moments may be used to deal with bodies in rotational equilibrium.

For systems in equilibrium, the lines of action of the non-parallel forces **pass through a common point**.

5 Work, Energy & Power

5.1 Work

Definition 5.1: Work

Work, W, on an object by a **constant force** $\vec{\mathbf{F}}$ is the product of the force and displacement $\vec{\mathbf{s}}$ in the direction of the force.

$$W = \vec{\mathbf{F}} \cdot \vec{\mathbf{s}} = (F \cos\theta)s$$

where F is the magnitude of force, s is the magnitude of displacement, and θ is the angle between the force and the displacement vectors

Work is a *scalar quantity* with S.I. unit joule (J), representing the transfer of energy to a system.

For several forces acting on an object, the **net work done** is the sum of the work done by each force separately, or the dot product between the net force and the displacement.

$$\begin{aligned}\sum W &= W_1 + W_2 + \cdots + W_n \\ &= \sum \vec{\mathbf{F}} \cdot \vec{\mathbf{s}} = \left(\sum F \cos\theta\right) s\end{aligned}$$

5.1.1 Work Done by Variable Forces

For a body displaced by a variable force, the net work done is the **integral of the force** with respect to displacement $\vec{\mathbf{s}}$.

$$\sum W = \int_{\vec{\mathbf{s}}_i}^{\vec{\mathbf{s}}_f} \sum \vec{\mathbf{F}}\, d\vec{\mathbf{s}}$$

In other words, the total work done on the body is equal to the **area under the force-displacement graph**.

5.2 Energy

Definition 5.2: Energy

Energy is the capacity to **do work**.

Energy is a *scalar quantity* with S.I. unit joule (J).

Definition 5.3: Principle of Conservation of Energy

The **principle of the conservation of energy** states that energy can neither be destroyed nor created. It can be transformed from one form to another and transferred from one body to another.

5.2.1 Kinetic Energy

Definition 5.4: Kinetic Energy

Kinetic energy is the energy that a body possesses solely due to its motion.

$$K = \frac{1}{2}mv^2$$

where K is the kinetic energy of a body of mass m moving with speed v.

Definition 5.5: Work-Energy Theorem

The **work-energy theorem** states that the work done by a resultant external force on a body is equal to the change in the kinetic energy of the body.

Proof. Consider an object of mass m moving through a displacement to the right under a net force $\sum \vec{\mathbf{F}}$, also directed to the right.

The net work done is

$$\sum W = \int_{x_i}^{x_f} \sum F \, dx$$

Using Newton's 2nd law, we substitute for the magnitude of the net force $\sum F = ma$, where m is constant,

$$\begin{aligned}\sum W &= \int_{x_i}^{x_f} ma \, dx = \int_{x_i}^{x_f} m\frac{dv}{dt} \, dx = \int_{x_i}^{x_f} m\frac{dx}{dt} dv = \int_{v_i}^{v_f} mv \, dv \\ &= \frac{1}{2}m{v_f}^2 - \frac{1}{2}m{v_i}^2\end{aligned}$$

The common quantity $\frac{1}{2}mv^2$ is the kinetic energy, K. As such, we can rewrite the equation as

$$\sum W = K_f - K_i = \Delta K$$

Hence, the net work done is equal to the change in kinetic energy, provided that the system *only undergoes translational motion*, where speed is the only change. □

5.2.2 Potential Energy

Definition 5.6: Potential Energy

Potential energy is the ability to do work as a result of the position, shape, or state of an object.

For a **conservative force** F, it is related to the potential energy U such that

$$F = -\frac{dU}{dx}$$

The gradient of the potential energy-displacement graph gives the magnitude of the conservative force. In addition, the force points in the direction of **decreasing** potential energy.

One of the most pertinent forms of potential energy is **gravitational potential energy**, further formalised in Chapter 7. To simplify, on Earth's surface, gravitational potential energy of an object is defined as

$$U_G = mgh$$

Consider an object at a certain height h_1 from the ground raised by a **constant force** F equal and opposite to the weight of the object. The object moves at a constant speed to a height h_2, such that no work is used to increase its kinetic energy.

The weight mg is equal to the force F. The work done by the force F is thus

$$\begin{aligned} W &= F \cdot s \\ &= mg(h_2 - h_1) \\ &= mgh_2 - mgh_1 \\ &= (U_G)_f - (U_G)_i = \Delta U_G \end{aligned}$$

Thus, the work done by a force in raising an object through a vertical height h at a constant speed results in a **change** of the gravitational potential energy of the object.

5.3 Conservation of Mechanical Energy

Definition 5.7: Mechanical Energy

Mechanical energy of a system is the sum of the kinetic energy and potential energy in the system.

Definition 5.8: Principle of Conservation of Mechanical Energy

The **principle of conservation of mechanical energy** states that in the absence of any net external force, the total mechanical energy of the system is conserved.

$$\Delta K + \Delta U = 0$$

This principle illustrates that while the kinetic and potential energies of a system may be converted or transformed into one another, the **sum of the energies is always conserved**.

However, most moving bodies experience dissipative forces. In these situations, the total mechanical energy is **not conserved**. The work done by dissipative forces is equal to the loss in mechanical energy, that is,

$$K_i + P_i + W_d = K_f + P_f$$

The work done by the dissipative forces W_d is negative, as the forces always act in opposite direction to the displacement.

5.4 Power

Definition 5.9: Power

Power is defined as work done per unit time.

Power is a **scalar** quantity with S.I. unit watt. For a constant force $\vec{\mathbf{F}}$ acting on an object moving at a velocity $\vec{\mathbf{v}}$, the instantaneous power, P, is defined as

$$P = \frac{dW}{dt} = \frac{d(\vec{\mathbf{F}} \cdot \vec{\mathbf{s}})}{dt} = \vec{\mathbf{F}} \cdot \left(\frac{d\vec{\mathbf{s}}}{dt}\right) = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}$$

The average power $\langle P \rangle$ is given by

$$\langle P \rangle = \frac{W}{t} = \frac{\vec{\mathbf{F}} \cdot \vec{\mathbf{s}}}{t}$$

5.4.1 Efficiency

Due to inevitable work done by dissipative forces, the energy conversion can never be 100%. The useful energy output will almost always certainly be less than the energy input. The efficiency of a machine in its conversion of energy from one form to another is defined as

$$\text{efficiency} = \frac{\text{useful energy output}}{\text{total energy input}} \times 100\%$$

6 Motion in a Circle

6.1 Radians

Consider an object moving in a circular arc. Let the radius of this circular path be r and the ball's travelled arc length be s with corresponding angular displacement θ.

The angle θ (in radians) is the ratio of the distance s along the circular arc subtended by θ divided by the radius r.

$$\theta = \frac{s}{r} \implies s = r\theta$$

In other words, a **radian** is the angle subtended at the centre of a circle by an arc of a length equal to the radius. If $s = r$, then θ is one radian. One radian is equal to $180°$.

6.2 Angular Velocity

Definition 6.1: Angular Velocity

Angular velocity, denoted by $\vec{\boldsymbol{\omega}}$, is the rate of change of angular displacement.

$$\vec{\boldsymbol{\omega}} = \frac{d\vec{\boldsymbol{\theta}}}{dt}$$

When the angular velocity is constant, the body is said to be undergoing **uniform circular motion**.

$$\omega = \frac{\text{angular displacement}}{\text{time taken}} = \frac{\theta}{t}$$

If a body undergoing uniform circular motion takes T seconds to complete a full revolution,

$$\omega = \frac{\theta}{t} = \frac{2\pi}{T} = 2\pi f$$

where f represents the frequency of the circular motion.

6.2.1 Relation to Linear Speed

From $s = r\theta$, the linear speed of an object can be determined by considering the displacement derivative.

$$\frac{ds}{dt} = \frac{d(r\theta)}{dt} = r\frac{d\theta}{dt} + \theta\frac{dr}{dt}$$

If r is a constant, then

$$\frac{ds}{dt} = r\frac{d\theta}{dt}$$

Converting these derivatives to their representative quantities, we obtain an equation relating linear speed and angular velocity. This relationship is only valid when ω is measured in radians per second.

$$v = r\omega$$

The direction of the linear velocity is tangential to the circular path. For an object rotating about an axis, every point on the object has the same angular velocity. The tangential speed of any point is proportional to its distance from the axis of rotation.

6.3 Centripetal Acceleration

Definition 6.2: Centripetal Acceleration

For a body in uniform circular motion, its acceleration is termed **centripetal acceleration** and is always perpendicular to the velocity of the object and points towards the centre of a circle.

$$a_c = \frac{v^2}{r} = v\omega = r\omega^2$$

Consider a body moving in a circular path of radius r with constant angular velocity ω. As it moves within an infinitesimally brief time interval dt, the magnitude of its linear velocity *does not change*, and the body moves an infinitesimal distance of arc length $r\,d\theta$.

Hence, the instantaneous *centripetal acceleration* of the body during this time interval is the derivative of linear velocity with respect to time, given by

$$\begin{aligned} a_c = \frac{dv}{dt} = \frac{d(r\omega)}{dt} &= \omega\frac{dr}{dt} + r\frac{d\omega}{dt} \\ &= v\omega = r\omega^2 = \frac{v^2}{r} \qquad \left(\because \frac{d\omega}{dt} = 0\right) \end{aligned}$$

Using trigonometric principles, it is possible to show that the direction of centripetal acceleration of the body undergoing uniform circular motion is towards the centre of the circular path. Centripetal acceleration is always perpendicular to the tangential velocity of the body.

6.4 Centripetal Force

Applying Newton's 2nd law to a body of mass m, the centripetal force that must act to produce the uniform circular motion is

$$F = mr\omega^2 = m\frac{v^2}{r} = mv\omega$$

For uniform circular motion, the centripetal force is constant in magnitude and changing in direction.

As the centripetal force acts perpendicularly to the motion, it only changes the direction of velocity, but not the speed of the body. Moreover, due to its direction, *centripetal force does no work.*

If the centripetal force producing the centripetal acceleration vanishes, the object does not continue to move in its circular path; instead, it will move along a tangential path.

To solve a problem involving circular motion,

1. Draw a free-body diagram for the problem.
2. Identify the center of the circular motion.
3. Choose a suitable coordinate system.
4. Resolve the forces, if necessary.

5. Apply Newton's Second Law to solve the problem, while ensuring that r is the radius of the circular path.

6.4.1 Don't Draw the Centripetal Force

Consider an object moving in a uniform circular motion at constant angular velocity ω along a curved path from A to B.

Without external forces, Newton's 1st law tells us it would move in a straight line at constant speed. Since it follows a curve, an external force must therefore pull it towards the circle's center (which is, in fact, the centripetal force acting perpendicular ot the object's motion to keep its speed constant).

According to Newton's 2nd law, this inward centripetal acceleration results from this force. However, it is important to note that the centripetal force *is not an extra force*; it represents the *net effect* of forces like tension and friction, and should not be drawn separately on free-body diagrams.

6.5 Turning Motion of Vehicles

Consider the following problem.

Example 6.1

A car of mass 750kg is taking a corner with a flat horizontal surface and a radius of 10m. If the lateral friction forces cannot exceed 1/10 of the weight of the vehicle, what is the maximum speed at which it can take the bend?

To begin, the frictional force by the road on the tyres is what provides the centripetal force.

$$f = m\frac{v^2}{r}$$

For maximum speed,

$$\begin{aligned} f_{\text{max}} &= m\frac{(v_{\text{max}})^2}{r} \\ \implies \frac{mg}{10} &= m\frac{(v_{\text{max}})^2}{10} \\ \implies v_{\text{max}} &= \sqrt{g} = 3.1m/s \end{aligned}$$

$\therefore$ the maximum speed at which the car can take the bend is 3.1m/s or 11km/h.

Consider the forces acting on a racing car on mass m travelling around a circular bend along a flat horizontal track.

Neglecting air resistance, each tyre experiences its own normal reaction and frictional force, but the forces acting on the front and rear tyres are combined so that there are simply two normal reactions, N_1 and N_2, and two frictional forces, f_1 and f_2.

Resolving forces horizontally,

$$f_1 + f_2 = ma$$

Resolving forces vertically,

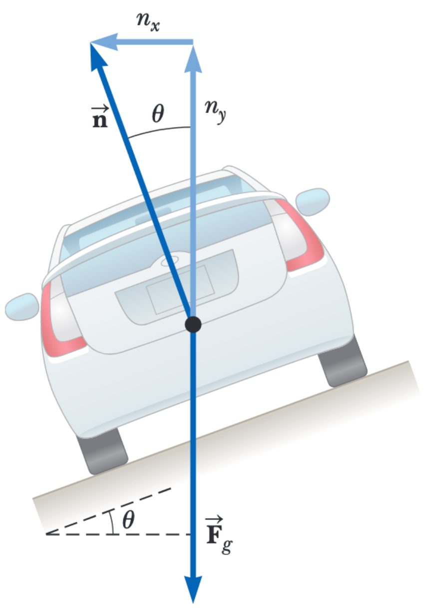
$$N_1 + N_2 = W$$

6.5.1 Banking Angle

As the tyres of the car can only provide a **maximum** friction, a car travelling too fast may not get around the bend. To reduce the possibility of skidding, the road can be **sloped** (or **banked**) so that the normal contact force of the road acting on the car provides a component directed to the centre of the circular path.

For a particular speed, this force will produce the exact centripetal force required. Otherwise, some lateral frictional force is needed.

Consider a car of mass m travelling on a slope of an angle θ between the road surface and the horizontal. Here, the horizontal component of the normal contact force N between the car and the road provides the centripetal force required to keep the car moving around the bend.



Resolving forces horizontally,

$$N_x = ma_c \implies N\sin\theta = m\frac{v^2}{r}$$

Resolving forces vertically,

$$N_y = W \implies N\cos\theta = mg$$

Dividing the first equation by the second,

$$\frac{N\sin\theta}{N\cos\theta} = m\frac{v^2}{r}\nabla \cdot mg \implies \tan\theta = \frac{v^2}{gr}$$

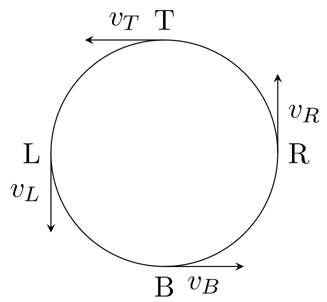
Hence, the optimal angle for the slope to be banked is given by

$$\boxed{\theta = \arctan\frac{v^2}{gr}}$$

6.6 Vertical Circular Motion

In vertical circular motion, the velocity of an object moving in a circular path differs as it travels between the bottom and the top of the loop.

Consider a body of mass m moving in a vertical circle of radius r on the end of a string.



At L and R, **tension of the string** provides the centripetal force.

At T and B, **weight and normal contact force** provide the centripetal force.

The speed changes as the mass moves upwards, as the **kinetic energy** increases as the expense of gravitational potential energy.

$$\Delta E_k = \Delta E_p$$
$$\frac{1}{2}m{v_B}^2 - \frac{1}{2}m{v_T}^2 = mg(2r)$$
$${v_B}^2 - {v_T}^2 = 4gr$$

At T,

$$F_c = N + W = mg + m\frac{v^2}{r}$$

At B,

$$F_c = N - W = mg - m\frac{v^2}{r}$$

7 Gravitational Fields

There he sat at rest, in the intellectual center: as the great solar orb shining with its own light, & diffusing his beamy influence, thro' the whole system of arts, & sciences. To him gravitated all the lesser lights, both regular planets, & extravagant comets of erudition, both at home, & abroad.

— William Stukely, *Memoirs of Sir Isaac Newton*

7.1 Newton's Law of Gravitation

Definition 7.1: Newton's Law of Gravitation

Newton's law of gravitation states that the gravitational attraction between two point masses m_1 and m_2 is directly proportional to the product of their masses and inversely proportional to the square of the separation r between their centres.

$$F = \frac{Gm_1m_2}{r^2}$$

where $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2} is the gravitational constant of proportionality.

Note that **point masses** have non-zero mass but no volume. If two objects are placed sufficiently far apart (where their dimensions become negligible compared to separation), they can be considered point masses.

The gravitational forces between two masses are equal and opposite (recall Newton's Third Law), and always act along the line joining the two point masses.

To show its **attractive** nature, it is typically written as

$$\vec{\mathbf{F}} = -\frac{Gm_1m_2}{r^2}\hat{\mathbf{r}}$$

The negative sign is ignored when only the magnitude of the force is required.

7.2 Gravitational Field

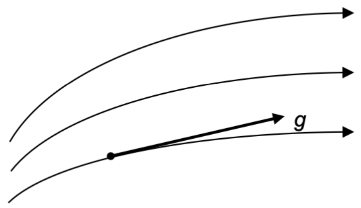
Every object with **mass** sets up a **gravitational field** in its surrounding space.

When two objects enter each other's gravitational fields, they will be attracted towards each other. When an object with mass is placed in a gravitational field, it will **experience a gravitational force** acting on it.

7.2.1 Field Direction

The direction of a field at a point is along a tangent to the field line at that point.

The density of the field lines at a point (number of lines per unit area) corresponds to the strength of the field at that point. A **denser arrangement** of field lines indicates **greater gravitational field strength**.



7.2.2 Gravitational Field Strength

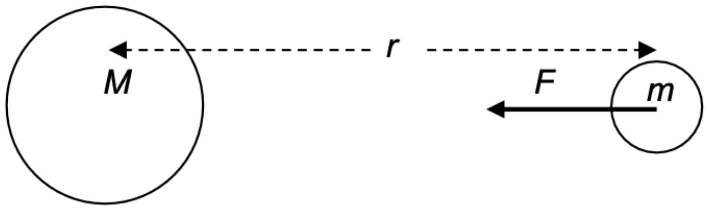
Definition 7.2: Gravitational Field Strength

The gravitational field strength g at a point of distance r from point mass M is defined as the **gravitational force per unit mass** exerted on a small test mass placed at that point.

$$\vec{\mathbf{g}} = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

The gravitational field strength is a **vector** quantity with S.I. unit Nkg^{-1}.

Consider two point masses M and m separated by a distance r.



From Newton's law of gravitation, the magnitude of the attractive gravitational force F acting on m due to M is

$$F = \frac{GMm}{r^2}$$

Since gravitational field strength is force per unit mass, the magnitude of g at m's position is equal to

$$g = \frac{F}{m}$$

From these two equations,

$$g = \frac{GMm}{r^2}\Big/m = \frac{GM}{r^2}$$

which is an expression for the **magnitude** of the gravitational field strength g at a point a distance r measured from a point mass M (it is M who **creates** the gravitational field).

A negative sign is used in the vector to show that the field strength points towards decreasing — $\hat{\mathbf{r}}$ is a unit vector in the **outward radial direction**.

At this junction, it is important to investigate the **constancy of field strength**, particularly for an affected point mass some minor distance away. The gravitational field strength g near the surface of the Earth is approximately constant at $9.81\mathrm{N\ kg}^{-1}$, which is also known as the acceleration of free fall with the value of $9.81\mathrm{m\ s}^{-2}$.

The approximate constancy should be appreciated by considering the value of g at height h above the surface, where h is small compared to the radius R of the Earth of mass M.

$$g = \frac{GM}{(h+R)^2} \approx \frac{GM}{R^2} \quad \text{for } h \ll R$$

This is illustrated through the example below.

Example 7.1

Calculate the gravitational field strength at a height of 1km above the Earth's surface. The radius of the Earth is 6400km.

At the surface, using $g = \frac{GM}{r^2}$,

$$9.81 = \frac{GM_E}{{R_E}^2} = \frac{GM_E}{(6400 \times 10^3)^2}$$

At a height of 1km, $r = 6401$km,

$$g_1 = \frac{GM_E}{r^2} = \frac{GM_E}{(6401 \times 10^3)^2} = 9.81 \times \frac{(6400 \times 10^3)^2}{(6401 \times 10^3)^2} = 9.81$$

7.3 Gravitational Potential Energy

Near the Earth's surface, gravitational potential energy takes on an elementary expression: it is simply denoted by $U = mgh$. However, when dealing with non-negligible distances away from Earth's surface, we must confront a more unwieldy formulation.

Definition 7.3: Gravitational Potential Energy

The **gravitational potential energy** of a mass at a point is defined as the **work done on the mass in moving it from infinity to that point**.

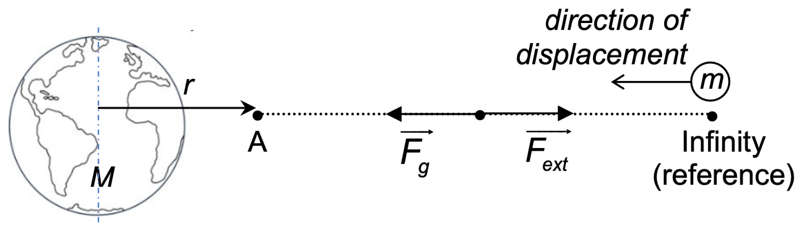
$$U = -\frac{GMm}{r}$$

Here, the work is done by an external force acting in the **opposite direction** to the gravitational attraction.

Alternatively, the gravitational potential energy of a mass at a point can also be understood as the **work done against gravity** in bringing the mass from infinity to that point.

To illustrate the meaning of the gravitational potential energy U, considering moving a mass m from infinity to a point A, at distance r from the centre of a mass M, at constant speed.

$\vec{\mathbf{F}}_g$ represents gravitational force acting on m by M, while $\vec{\mathbf{F}}_{\text{ext}}$ represents the force acting on m by an external agent such that $|\vec{\mathbf{F}}_{\text{ext}}| = |\vec{\mathbf{F}}_g|$.



Mathematically, in combination with Newton's 3rd law, the definition can be written as

$$\begin{aligned} U &= \int_{\infty}^{r} \vec{\mathbf{F}}_{\text{ext}}\, dr \\ &= \int_{\infty}^{r} -\vec{\mathbf{F}}_g\, dr \\ &= \int_{\infty}^{r} \left(\frac{GMm}{r^2}\right) dr \\ &= \left[-\frac{GMm}{r}\right]_{\infty}^{r} = -\frac{GMm}{r} \end{aligned}$$

As iterated in Chapter 4, the gravitational force is the derivative of U with respect to r.

$$F = -\frac{dU}{dr}$$

7.3.1 Gravitational Potential

Definition 7.4: Gravitational Potential

The **gravitational potential** at a point is defined as the work done per unit mass (by an external force) in bringing a small test mass from infinity to that point.

$$\phi = \frac{U}{m} = -\frac{GM}{r}$$

It is a **scalar** quantity with S.I. unit Jkg^{-1}.

The gravitational potential energy of a mass depends on both the value of the mass as well as the position of the mass.

The gravitational potential is an energy-related quantity solely dependent on the position in a gravitational field. Moreover, the gravitational field strength g can be said to be the negative gradient of the gravitational potential ϕ with respect to r.

$$g = -\frac{d\phi}{dr}$$

7.4 Energy Considerations

7.4.1 Total Energy

With the formal definition of gravitational potential energy, we are able to analyse the energy for a body of mass m orbiting around a larger body of mass M.

Since gravitational force provides centripetal force,

$$\begin{aligned}\sum F = ma &\implies \frac{GMm}{r^2} = \frac{mv^2}{r} \\ &\implies \frac{1}{2}mv^2 = \frac{GMm}{2r}\end{aligned}$$

Hence, for gravitation,

$$E = K + U = \frac{GMm}{2r} + \left(-\frac{GMm}{r}\right) = -\frac{GMm}{2r}$$

The above holds true for circular orbits. For elliptical orbits, simply take r to be the semi-major axis length.

7.4.2 Escape Velocity

As an object travels indefinitely away from a body, $r \to \infty$, hence $K \to 0$, $U \to 0$, and $E \to 0$. As such, at infinity, we take the kinetic energy and gravitational potential energy **to be zero**.

Since the object-Earth system is isolated, calculating the minimum escape velocity v,

$$\begin{aligned}\Delta U = -\Delta K &\implies 0 - \left(-\frac{GMm}{r}\right) = -\left(0 - \frac{1}{2}mv^2\right) \\ &\implies \frac{GMm}{r} = \frac{1}{2}mv^2 \\ &\implies \boxed{v = \sqrt{\frac{2GM}{r}}}\end{aligned}$$

If the initial velocity is greater than the minimum escape velocity required, the object will have some residual kinetic energy as $r \to \infty$.

7.5 Kepler's Third Law

Take the situation of the circular motion of our Moon (of mass m) assumed to be circulating about our Earth (of mass M) in a circular orbit of radius r.

This simplifies the case to a **gravitational one-body problem**, where one object orbits about a point under the influence of gravity.

The **gravitational force** acting on the Moon by the Earth provides the centripetal force, giving acceleration $a_c = r\omega^2$.

$$\begin{aligned}
\sum F = ma \implies \frac{GMm}{r^2} &= mr\omega^2 \\
\implies \omega^2 &= \frac{GM}{r^3} \\
\implies \left(\frac{2\pi}{T}\right)^2 &= \frac{GM}{r^3} \\
\implies r^3 &= \frac{GM}{4\pi^2}T^2 \\
\implies &\boxed{r^3 \propto T^2}
\end{aligned}$$

This is known as **Kepler's Third Law**, which states that the square of a body's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

7.6 Geostationary Orbits

Geostationary orbits are circular orbits made by satellites about the Earth's axis such that these satellites would **appear stationary when observed from a point on the surface of the Earth**. This allows for the satellite to provide permanent coverage of a given wide area.

1. They must move **west to east** (moves in the same sense of self-rotation of the Earth about its own axis).
2. The satellite's **orbital period must be equal to 24 hours** (so that it is the same as Earth's rotational period about its own axis).
3. The geostationary orbit is an equatorial **equatorial orbit** (an orbit in the plane of the equator) — for the satellite to remain in a fixed relative position above the Earth's surface, its axis of rotation has to coincide with Earth's axis of rotation.

Part III

Thermal Physics

8 Temperature and Ideal Gases

Every mathematician knows it is impossible to understand an elementary course in thermodynamics.

— Vladimir Igorevich Arnold

8.1 Zeroth Law of Thermodynamics

Temperature is a measure of the degree of hotness of an object. Thermal energy moves from objects of higher temperatures to objects of lower temperatures. Note that temperature does not measure an object's thermal energy.

Heat is **energy** that flows from a higher-temperature object to a lower-temperature object due to differences in temperature.

Definition 8.1: Thermal Equilibrium

When two objects in thermal contact has no *net heat transfer* between them, they are said to be in **thermal equilibrium** and are at the same temperature.

Definition 8.2: Zeroth Law of Thermodynamics

The **zeroth law of thermodynamics** states that if two objects A and B are separately in thermal equilibrium with object C, then objects A and B must be in thermal equilibrium with each other.

This forms the basis for the measurement of temperature — the object C named above can function as a measuring device known as a *thermometer*.

8.2 Temperature Scales

Thermometers measure temperature based on a physical property that changes with temperature.

8.2.1 Empirical Temperature Scale

An **empirical temperature scale** is a scale of temperature that is based on the variation with temperature of a property of a substance, *assuming* that the property varies linearly with temperature.

To set up an empirical temperature scale,

1. Choose an appropriate thermometric property, which can be one of the following:
    - volume of fixed mass of liquid
    - resistance of a metal
    - pressure of a fixed mass of gas at constant temperature
    - electromotive force produced between junctions of dissimilar metals at different temperatures

2. Select two fixed temperature points; an upper point and a lower point to calibrate the thermometer.
3. Calibrate the thermometer by placing the thermometer at the two points, and record the thermometric quantity, letting the value at the lower point be X_0 and that at the upper point be X_100.

With these two values, and holding the assumption that there exists a linear relationship between these two points, the temperature θ of any subsequent medium can be calculated as shown.

$$\theta = \frac{X_\theta - X_0}{X_{100} - X_0} \times 100$$

8.2.2 Absolute Temperature Scale

The **absolute temperature scale** (also thermodynamic temperature scale) does not depend on the thermometric property of any particular substance, and has absolute zero and the triple point of water as fixed points.

- **Absolute zero**, defined as $-273.15°$C or 0K, is the temperature at which all substances have **minimum internal energy**.
- The **triple point of water** is the particular temperature and pressure (273.16K, 4.58mmHg) at which the three states of water can co-exist in equilibrium.

In other words, the absolute temperature scale is not based on any thermometric property, but rather on the theoretical efficiency of a perfectly reversible heat engine.

8.3 Ideal Gas Law

Definition 8.3: Ideal Gas

An **ideal gas** is a hypothetical gas that obeys the equation of state

$$PV = nRT$$

perfectly at all pressures P, volumes V, amounts of substances n, and temperatures T.

For an ideal gas with volume V, pressure P, and temperature T, experiments have shown that

- by Charles' law, $V \propto T$ when P is constant;
- by Boyle's law, $P \propto 1/V$, when T is constant;
- by Gay-Lussac's law, $P \propto T$, when V is constant.

These laws give the **ideal gas law** where n is the number of **moles** of gas, and $R = 8.31\,\text{J K/mol}$ is the **universal gas constant**.

$$PV = nRT$$

This can be rewritten as, where N is the number of **molecules** of gas, and $k = R/N_A = 1.38 \times 10^{-23}\,\text{J/K}$ is **Boltzmann's constant**.

$$PV = NkT$$

Avogadro's Number

One mole is the amount of substance containing 6.02×10^{23} particles.

The mathematical relationship between n and N is

$$N = nN_A$$

where $N_A = 6.02 \times 10^{23}$ is **Avogadro's number**.

8.4 Kinetic Theory of Gases

In the kinetic theory of gases, the **assumptions imposed on ideal gases** are:

- all gases consist of a very large number of atoms or molecules;
- the atoms or molecules behave as if they are hard, perfectly elastic, identical spheres;
- the molecules are in constant, random motion, and obey Newton's laws of motion;
- there are no forces of attraction or repulsion between atoms or molecules unless they are in collision with each other or with the walls of the container (*therefore potential energy is zero*);
- total volume of atoms or molecules is negligible compared with the volume of the container;
- time of collisions is negligible compared with time between collisions.

8.4.1 Molecular Model

Consider a cubical box of sides l, containing N molecules of a gas, where the mass of each molecule is m.

By considering the *elastic collision of one molecule* (with x-velocity c_x) *with a wall*, the change in momentum of the molecule due to the collision is

$$\Delta p_i = -2mc_{xi} \implies \Delta p_{\text{wall}} = 2mc_{xi}$$

The next time the same molecule collides with the same wall, the molecule would have travelled a total distance of $2l$ in the x direction. Hence, the time Δt between collisions is

$$\Delta t = \frac{2l}{c_{xi}}$$

Therefore, the rate of change of momentum for the wall due to one molecule is

$$F_i = \frac{\Delta p_{\text{wall}}}{\Delta t} = (2mc_{xi})\left(\frac{c_{xi}}{2l}\right) = \frac{m{c_{xi}}^2}{l}$$

From Newton's second law,

$$F_{\text{wall}} = \sum_{i=1}^{N} \frac{m{c_{xi}}^2}{l} = \frac{m}{l}\sum_{i=1}^{N} {c_{xi}}^2$$

From the definition of **mean square speed**,

$$\langle {c_x}^2 \rangle = \frac{{c_{x1}}^2 + {c_{x2}}^2 + \cdots + {c_{xN}}^2}{N} \implies F_{\text{wall}} = \frac{Nm}{l}\langle {c_x}^2 \rangle$$

$$\implies P_{\text{wall}} = \frac{F_{\text{wall}}}{l^2} = \frac{Nm}{V}\langle {c_x}^2 \rangle$$

Applying Pythagoras' theorem to the 3-dimensional velocity vector for any molecule leads to

$$c^2 = {c_x}^2 + {c_y}^2 + {c_z}^2$$
$$\langle c^2 \rangle = \langle {c_x}^2 \rangle + \langle {c_y}^2 \rangle + \langle {c_z}^2 \rangle$$

However, we assume that the motion of particles is isotropic in the kinetic theory, such that $\langle {c_x}^2 \rangle = \langle {c_y}^2 \rangle = \langle {c_z}^2 \rangle$. Therefore, $\langle {c_x}^2 \rangle = \langle c^2 \rangle / 3$.

Using this expression, we can find the total force and pressure exerted on the wall.

$$F = \frac{1}{3}\frac{Nm}{l}\langle c^2 \rangle \implies \boxed{PV = \frac{1}{3}Nm\langle c^2 \rangle}$$

8.4.2 Average Molecular Kinetic Energy

Notice that we obtain two forms of the expression PV. This allows us to relate average **translational molecular kinetic energy** to temperature.

$$PV = \frac{Nm}{3}\langle c^2 \rangle = NkT \implies T = \frac{2}{3k}\left(\frac{1}{2}m\langle c^2 \rangle\right)$$

$$\implies \boxed{\frac{1}{2}m\langle c^2 \rangle = \frac{3}{2}kT}$$

8.4.3 Representations of Ideal Gas Law

The kinetic theory of gases helps to spawn many other equivalent representations of the ideal gas law.

$$PV = nRT = NkT = \frac{1}{3}Nm\langle c^2 \rangle$$

To summarise, for a pressure P, volume V, amount of moles of gas n, amount of molecules of gas N, mass per molecule m, total mass of gas M, and molar mass of gas M_m, we have the three following equations.

Given density,

$$PV = \frac{1}{3}Nm\langle c^2 \rangle \implies PV = \frac{1}{3}M\langle c^2 \rangle \ \ (\because M = Nm)$$

$$\implies c_{\text{rms}} = \sqrt{\langle c^2 \rangle} = \sqrt{\frac{3PV}{M}} = \boxed{\sqrt{\frac{3P}{\rho}}}$$

Given molar mass,

$$
\begin{aligned}
PV = nRT &\implies \frac{1}{3}M\langle c^2\rangle = \frac{M}{M_m}RT \\
&\implies \langle c^2\rangle = \frac{3RT}{M_m} \\
&\implies c_{\text{rms}} = \boxed{\sqrt{\frac{3RT}{M_m}}}
\end{aligned}
$$

Given molecular mass,

$$
\begin{aligned}
PV = NkT &\implies \frac{1}{3}Nm\langle c^2\rangle = NkT \\
&\implies c_{\text{rms}} = \boxed{\sqrt{\frac{3kT}{m}}}
\end{aligned}
$$

9 First Law of Thermodynamics

9.1 Internal Energy

For a system with N molecules, the **state** of the system is defined by its pressure P, volume V, and thermodynamic temperature T.

Definition 9.1: Internal Energy

The **internal energy of a system** is the **sum** of the **kinetic energy** due to the random motion of the molecules, and the **potential energy** due to the intermolecular forces of attraction.

$$U_i = \sum K + \sum U_p = \sum K \quad (\because U_p = 0 \text{ for ideal gases})$$

For a stationary system, the internal kinetic energy of all the molecules is the sum of their **random kinetic energy**.

The internal potential energy of the molecules is due to the **work done to move them further apart against intermolecular attraction**. In ideal gases, we assume that there is no attraction between molecules, hence internal potential energy is considered to be **zero**.

9.1.1 Increase in Thermodynamic Temperature

As mentioned, for an **ideal gas**, its molecules have **no intermolecular attraction** (thus *no random potential energy*), meaning that all internal energy is due to its random kinetic energy. Recall that

$$\langle K \rangle = \frac{1}{2}m\langle c^2 \rangle = \frac{3}{2}kT$$

The internal energy U of an ideal gas with N molecules is thus

$$U = \sum K = N\langle K \rangle = \frac{3}{2}NkT \implies U \propto T$$

The internal energy of a fixed mass of ideal gas is **directly proportional** to its thermodynamic temperature. A rise in temperature of an ideal gas indicates **a rise in the average random kinetic energy** of the molecules.

One might also note that U is *simply a function of state*, as described later on in the isotherm, a state can be described by a unique pair of coordinates (P_i, V_i). As such, each $U = \frac{3}{2}PV$ *can be tied to a distinct point* on the P-V graph (a.k.a. a state).

9.2 Heat Capacity

The **heat capacity** C of an object is the thermal energy Q per unit temperature change ΔT

$$Q = C\Delta T$$

Note that $C = mc$, where c is the specific heat capacity as defined below. The unit of C is J K^{-1}.

Symbol	Meaning	Formula
ΔU	Positive $\rightarrow$ increase in internal energy Negative $\rightarrow$ decrease in internal energy	For an ideal gas, $\Delta U = \frac{3}{2}Nk\Delta T$
Q	Positive $\rightarrow$ heat **supplied to** the system Negative $\rightarrow$ heat **removed from** the system	$Q = mc\Delta T$ or $Q = mL$
W	Positive $\rightarrow$ volume of system **decreases** Negative $\rightarrow$ volume of system **increases**	$W = -\int_{V_1}^{V_2} p\,dV$

9.2.1 Specific Heat Capacity

Definition 9.2: Specific Heat Capacity

The **specific heat capacity** c of a subsatnce is defined as the thermal energy per unit mass per unit temperature change.

If the thermal energy Q supplied to a mass m of the substance changes its temperature by ΔT, then

$$c = \frac{Q}{m\Delta T} \implies Q = mc\Delta T$$

9.2.2 Specific Latent Heat

Definition 9.3: Specific Latent Heat

The **specific latent heat** of a substance is defined as the thermal energy per unit mass required to change its state at constant temperature.

From solid to liquid, it is latent heat of fusion, and latent heat of solidification for the reverse. They occur at the substance's melting point, with constant pressure.

From liquid to gas, it is latent heat of vaporisation, and latent heat of condensation for the reverse. They occur at the substance's boiling point, with constant pressure.

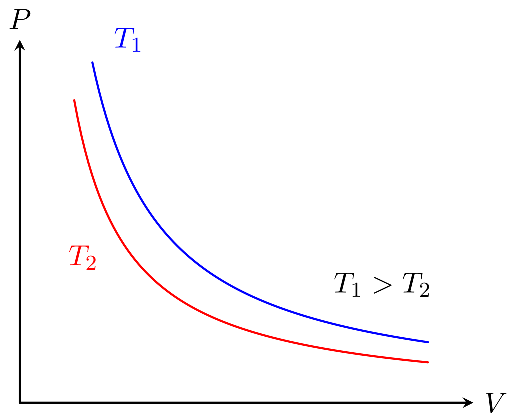
If the thermal energy Q supplied to a substance causes a change in state of a mass m of the substance, then the specific latent heat of the substance L is expressed as

$$L = \frac{Q}{M} \implies Q = mL$$

9.3 First Law of Thermodynamics

Definition 9.4: First Law of Thermodynamics

The **first law of thermodynamics** states that the **increase in internal energy** of a system is the **sum** of the **heat supplied to** the system and the **work done on** the system.



9.3.1 Isotherm

The state of a system can be represented by a point in the P/V graph.

Using the ideal gas equation $PV = nRT$, the temperature is of a specific value once there are specific values for P and V.

At constant temperature, the product PV is a constant, thus the P/V is a hyperbola called the **isotherm**.

From the above graph, since the larger the value of PV, the higher T is, hence $T_2 > T_1$.

9.3.2 Types of Processes

An **adiabatic process** is a (fast) process during which **no heat** is supplied to or removed from the system.

- $Q = 0 \implies \Delta U = W$

An **isobaric process** is a process that occurs at **constant pressure**.

- $W = -P\Delta V$
- Constant pressure means the environment is not changing — final state is on a different isotherm, with changes in T and V.

An **isovolumetric** (or **isochoric**) **process** is a process that occurs at **constant volume**.

- $V_f = V_i \implies W = 0 \implies \Delta U = Q$.
- Usually means the system is confined within a rigid container, with zero work done by and on the system — final state is on a different isotherm, with changes in T and P.

An **isothermal process** is a process that occurs at **constant temperature**.

- $\Delta U = 0 \implies Q = -W$
- The $P - V$ graph is a hyperbolic curve called an **isotherm**; constant temperature → constant total random kinetic energy → **constant internal energy**.
- Both P and V change, but the product PV is constant.

For cyclic processes, the graph maps the transitions between the thermodynamic states of the gas. At the end of 1 cycle, the gas **returns to its original thermodynamic state**, thus the overall change in internal energy across the entire cyclic process $\Delta U = 0$.

Part IV

Oscillations and Waves

10 Oscillations

Definition 10.1: Oscillation

An **oscillation** is a periodic to-and-fro motion of an object between two limits.

There are three types of oscillations: free oscillations, damped oscillations, and forced oscillations.

10.1 Free Oscillations

Definition 10.2: Free Oscillation

When an object undergoes **free oscillation**, it oscillates with **no energy gain or less**. The object oscillates with **constant amplitude**, as there are no external forces acting on it.

In the absence of any drag force, an oscillatory motion can be graphically represented by a **sinusoidal curve**.

As such, we can mathematically describe its **displacement** (x) with respect to time (t) as the following general question, where we are usually concerned with the particular solutions in which $\phi = 0°$ or $\phi = 90°$.

$$x = x_o \sin(\omega t) \qquad \text{or} \qquad x = x_o \cos(\omega t)$$

For the sine curve equation, the object passes the equilibrium position at $t = 0$. For the cosine curve equation, the object is at the extreme position at $t = 0$.

The following physical quantities are used to characterise oscillatory motions.

Definition 10.3: Amplitude (Oscillations)

The **amplitude**, x_0, is the **maximum displacement** of the oscillating object from the **equilibrium position**.

Definition 10.4: Displacement (Oscillations)

The **displacement**, x, is the distance of the oscillating object from its equilibrium position in a stated direction.

Definition 10.5: Period (Oscillations)

The **period**, T, is the time taken for one complete oscillation.

Definition 10.6: Frequency (Oscillations)

The **frequency**, f, the number of complete to-and-fro cycles per unit time made by the oscillating object.

$$f = \frac{1}{T}$$

Definition 10.7: Angular Frequency

The **angular frequency**, ω, is the constant which characterises the particular simple harmonic oscillator and is related to its natural frequency.

$$\omega = 2\pi f = \frac{2\pi}{T}$$

10.1.1 Phase

Apart from the quantities mentioned above, there are two additional quantities of importance. The **phase** is an angle which gives a measure of the fraction of a cycle that has been completed by an oscillating particle or by a wave.

- The phase of the motion of a particle is the quantity $(\omega t + \phi)$.
  - At the stage when $1/4$ cycle is completed, the phase is $\pi/2$ radians.
  - At the stage when $1/2$ cycle is completed, the phase is π radians.
  - At the stage when $3/4$ cycle is completed, the phase is $3\pi/2$ radians.
- A graph of $x = x_o \cos(\omega t + \phi)$ is the graph of $x_o \cos \omega t$ displaced to the left by a time interval $\frac{\phi}{\omega}$, with the plus sign indicating the motion leads by time $\frac{\phi}{\omega}$.
- A graph of $x = x_o \cos(\omega t - \phi)$ is the graph of $x_o \cos \omega t$ displaced to the right by a time interval $\frac{\phi}{\omega}$, with the negative sign indicating the motion lags behind by time $\frac{\phi}{\omega}$.

The **phase difference** between two oscillations is a measure of how much one oscillation is out of step with another.

- If two oscillations are in step with one another, they are said to be **in phase** with one another (phase difference $= 0$)
- Two oscillations are said to be **in anti-phase** (phase difference $= \pi$) when the displacement of each oscillation is completely opposite to each other.

10.2 Simple Harmonic Motion

Definition 10.8: Simple Harmonic Motion

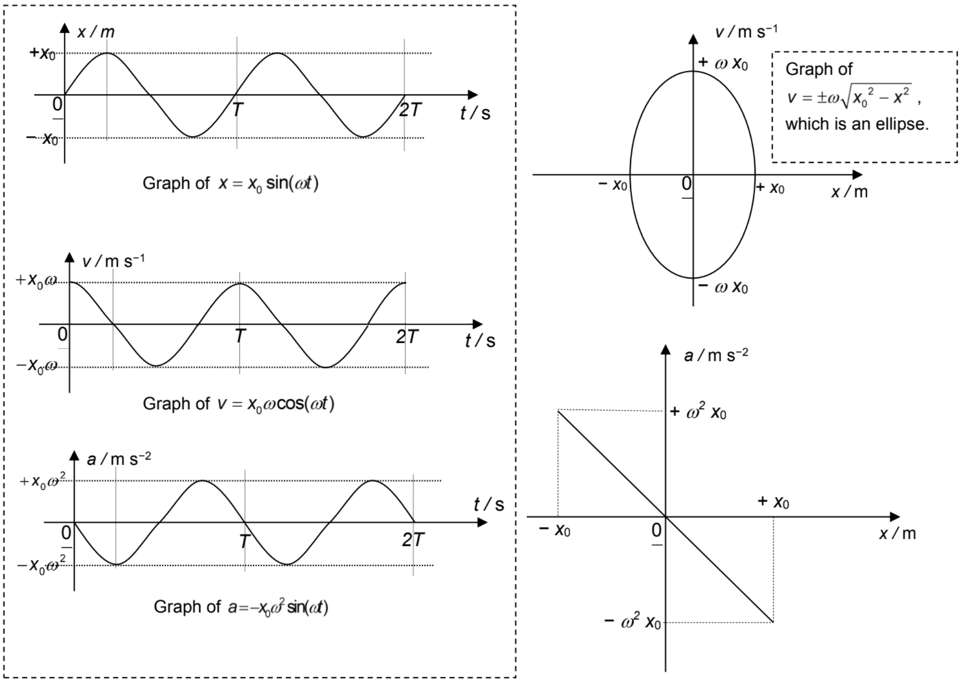
Simple harmonic motion is defined as the oscillatory motion of a particle whose acceleration is directly proportional to its displacement from a fixed point and this acceleration is always in opposite direction to its displacement.

$$a = -\omega^2 x$$

where x is the displacement from the equilibrium position, and ω^2 is the constant of proportionality.

The constant is a squared quantity to ensure it is always positive — as such, the acceleration a is always opposite to the direction of displacement x.

Note that the acceleration of a body undergoing simple harmonic motion is always directed towards the equilibrium position — the position at which **no net force** acts on the oscillating object.



To prove that $x = x_0 \sin \omega t$ is a particular solution to the defining equation $a = -\omega^2 x$,

$$v = \frac{dx}{dt} = \omega x_0 \cos \omega t$$

$$a = \frac{dv}{dt} = -\omega^2 (x_0 \sin \omega t) = -\omega^2 x$$

Re-examining, we can obtain the velocity-time relationship by

$$v = \frac{dx}{dt} = x_0 \omega \cos \omega t$$
$$\implies v = v_0 \cos \omega t$$

The velocity can also be more explicitly given by

$$v = \pm \omega \sqrt{({x_o}^2 - x^2)}$$

Similarly, we can find the acceleration-time relationship.

$$a = \frac{dv}{dt} = -v_0 \omega \sin \omega t$$
$$\implies a = -a_0 \sin \omega t$$

10.2.1 Graphical Illustration

Below are the graphs to describe the equations for displacement, velocity, and acceleration during free oscillatory motion.

The graphs of x-v and v-a have a phase difference of $\frac{\pi}{2}$. In other words, the displacement-time graph and acceleration-time graph are **in anti-phase** with each other.

10.2.2 Energy Expressions

The principle of conservation of mechanical energy states that the total energy in an isolated system is constant.

For an oscillator in simple harmonic motion, its **total energy** is the sum of its **kinetic** and **potential** energy.

$$E_{\text{total}} = E_{\text{kinetic}} + E_{\text{potential}}$$

If $x = x_0 \sin \omega t$, then $v = \omega\sqrt{({x_o}^2 - x^2)}$, thus allowing us to express E_k in terms of t,

$$\begin{aligned} E_k = \frac{1}{2}mv^2 &= \frac{1}{2}m\omega^2({x_0}^2 - x^2) \\ &= \frac{1}{2}m\omega^2{x_0}^2\cos^2(\omega t) \end{aligned}$$

The **maximum kinetic energy** of an oscillator in simple harmonic motion is given by

$$E_k = \frac{1}{2}m\omega^2{x_0}^2$$

and occurs when the oscillator is at its **equilibrium position**.

When the kinetic energy of an oscillating object is at its maximum, its potential energy must be at its minimum, since the total energy is constant.

If we define the minimum E_p to be 0, then

$$E_T = E_k = E_p$$

Thus, in terms of t,

$$\begin{aligned} \frac{1}{2}m\omega^2{x_0}^2 &= \frac{1}{2}m\omega^2{x_0}^2\cos^2(\omega t) + E_p \\ E_p &= \frac{1}{2}m\omega^2{x_0}^2\sin^2(\omega t) \end{aligned}$$

or, in terms of x,

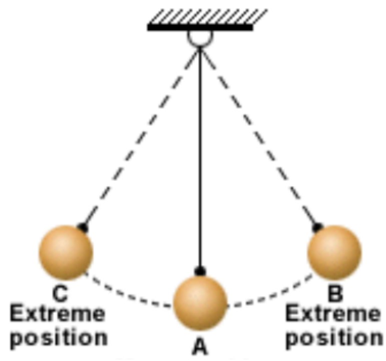
$$\begin{aligned} \frac{1}{2}m\omega^2{x_0}^2 &= \frac{1}{2}m\omega^2({x_0}^2 - x^2) + E_p \\ E_p &= \frac{1}{2}m\omega^2x^2 \end{aligned}$$

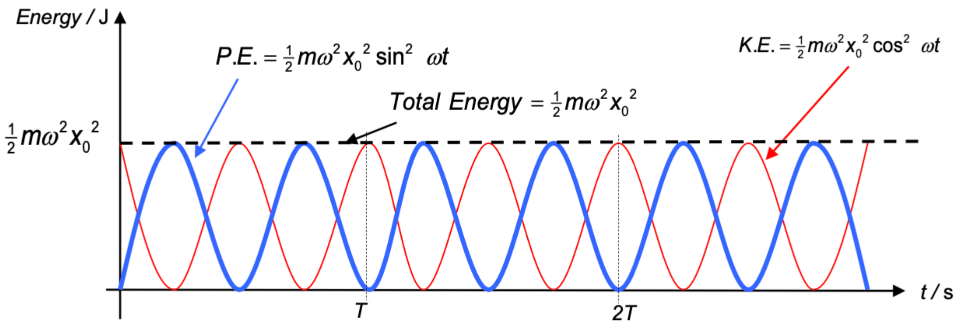
10.2.3 Energy Changes during Oscillation

Consider one cycle of the motion of a simple pendulum in the absence of energy losses to the surroundings, where the pendulum moves A → B → A → C → A.

At extreme positions B and C, where the bob is at maximum displacement, it has **maximum potential energy** and **minimum kinetic energy**. At the equilibrium position A, where the bob has zero displacement, it has **minimum potential energy** and **maximum kinetic energy**. In between the equilibrium position and the extreme position, the bob has a combination of potential and kinetic energy.

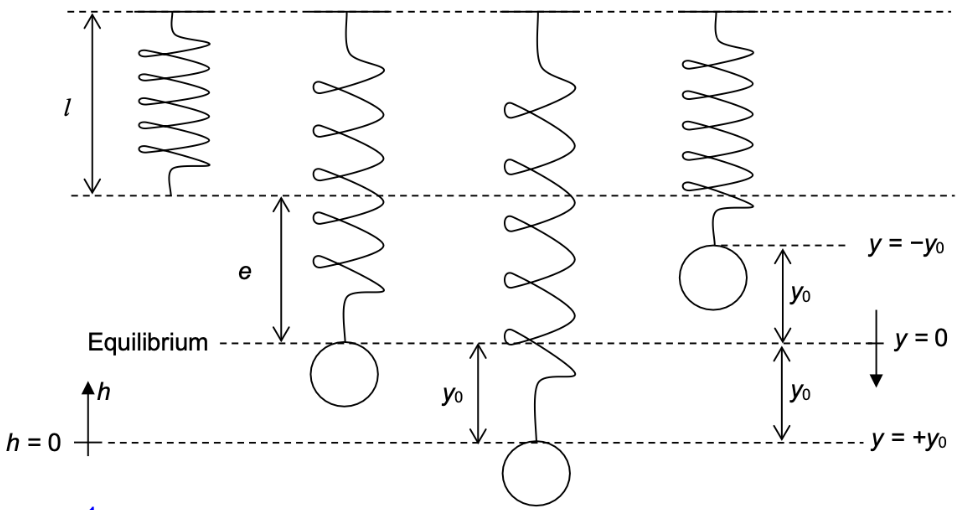
Since simple harmonic motion is assumed to take place in a closed system, the **total energy remains constant**.





10.2.4 Vertical Spring-Mass System

A vertically-oscillating spring-mass system interchanges three types of energies — kinetic energy, gravitational potential energy, and elastic potential energy.



Assuming that the spring obeys Hooke's law, and that $y_0 \leq e$,

The total energy in the isolated system is the sum of all three energies. In equilibrium, the mass experiences gravitational weight mg and the restoring force of the spring kx. This means that $kx = mg$.

Position	KE	GPE	EPE
Highest point, $y = -y_0$	0	$2mgy_0$	$\frac{1}{2}k(e-y_0)^2$
Equilibrium position, $y = 0$	$\frac{1}{2}m\omega^2{y_0}^2$	mgy_0	$\frac{1}{2}ke^2$
Lowest point, $y = +y_0$	0	0	$\frac{1}{2}k(e+y_0)^2$
General displacement y	$\frac{1}{2}m\omega^2({y_0}^2-y^2)$	$mg(y_0-y)$	$\frac{1}{2}k(e+y)^2$

10.3 Damped Oscillations

Definition 10.9: Damping

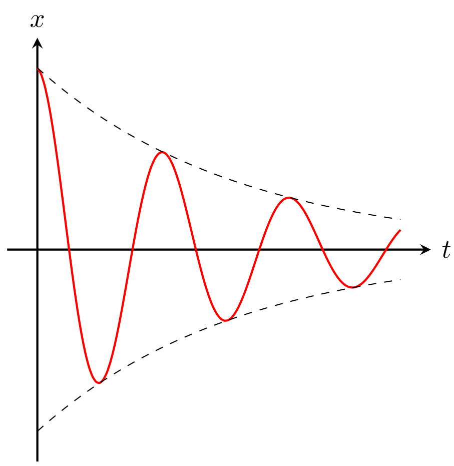
Damping is a process where energy is taken from an oscillating system as a result of dissipative forces.

Definition 10.10: Damped Oscillation

Damped oscillation occurs when there is a **continuous dissipation of energy to the surroundings** such that the total energy in the system decreases with time, hence the **amplitude of the motion progressively decreases with time**.

There are three degrees of damping.

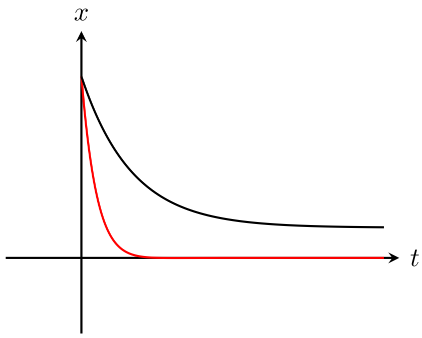
Light damping, where the object undergoes a number of complete oscillations with the amplitude of vibration decreasing exponentially with time. One example is a vertical simple spring-mass system in air or in a low-viscosity liquid.



Displacement-time graph of a lightly damped oscillator.

Critical damping, where no oscillations occur. The displacement is brought to zero in the *shortest possible time*. An example is a car with a suspension system which reduces discomfort on bumpy roads.

Heavy damping, where no oscillations occur about the equilibrium position when the damping force increases beyond the point of critical damping. The system *takes a longer time to return to the equilibrium position* compared to the critically damped system.



Displacement-time graph of a critically damped oscillator (red) and a heavily damped oscillator (black).

10.3.1 Car Suspension Systems

The degree of damping a mechanical system is important — a suspension system of a car should ensure that passengers are comfortable when moving on bumpy roads.

If the suspension is only **lightly damped**, passengers would be thrown up and down since the suspension system would take quite long to stop oscillating. This might also damage the car.

A good suspension system is one where shock absorbers on a car **critically damp** the suspension of the vehicle.

10.4 Forced Oscillations

Definition 10.11: Forced Oscillations

Forced oscillations are caused by the continual input of energy by external applied force to an oscillating system to compensate the loss due to damping in order to maintain the amplitude of the oscillation.

The system then oscillates at the frequency of the external periodic force.

10.4.1 Resonance

When the driving frequency f equals the natural frequency of a system f_0, the amplitude of the oscillation will be a **maximum**. This is called **resonance**.

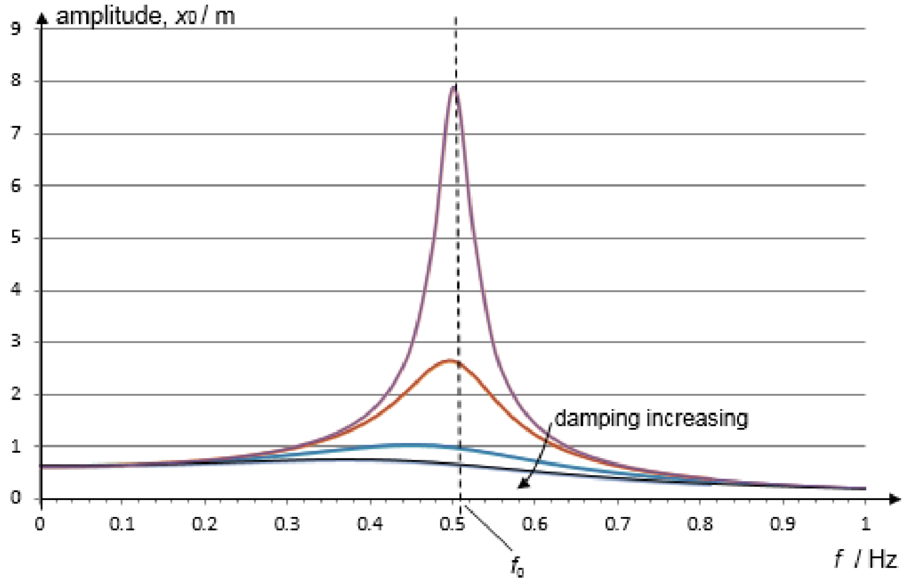
Definition 10.12: Resonance

Resonance occurs when the **resulting amplitude of the system becomes a maximum** when the driving frequency of the external driving force equals the natural frequency of the system.

Useful applications of include include radio and musical instruments, wheras undesirable effects of resonance include the shattering of glass (with the correct frequency and amplification) and resonance in rigid structures.

10.4.2 Frequency Response Graph

The graph below shows how the amplitude x_0 of a forced oscillation depends on the driving frequency f when the damping of the system is light, and when the damping of the system is heavy.



For a forced oscillation, when conditions are **steady**, one observes the following:

- the amplitude of a forced oscillation depends upon
  - the damping of the system; and
  - the relative values of the driving frequency f and the natural frequency f_0 of free oscillation (i.e. how far f is from f_0)
- the vibrations with largest amplitude (i.e. resonance) occur when f is equal to f_0

As damping increases,

- the response is **less sharp** (with a lower and flatter peak and smaller amplitudes at all frequencies), and
- the maximum amplitude is reached when the driving frequency is **slightly less than** the natural frequency

11 Wave Motion

11.1 Waves

11.1.1 Types of Waves

Definition 11.1: Progressive Wave

A **progressive wave** is a wave in which energy is carried from out point to another by means of vibrations or oscillations within the wave, without transporting matter.

Definition 11.2: Transverse Wave

A **transverse wave** is a wave with vibrations perpendicular to the direction of transfer of the wave's energy.

Definition 11.3: Longitudinal Wave

A **longitudinal wave** is a wave with vibrations parallel to the direction of transfer of the wave's energy.

Definition 11.4: Mechanical Wave

A **mechanical wave** is a wave that requires a medium for its transmission.

Definition 11.5: Electromagnetic Wave

A **electromagnetic wave** is a wave that consists of oscillating electromagnetic fields that are perpendicular to each other and to the direction of transfer of the wave's energy. They *do not require a medium* for transmission.

11.1.2 Electromagnetic Spectrum

Name	Wavelength	Uses
Radio waves	>10cm	Communications, radioastronomy
Microwaves	1mm to 10cm	Communications, cooking
Infrared	700nm to 1mm	Satellite surveying, TV controls
Visible light	400nm to 700nm	Sight, communication
Ultraviolet	1nm to 400nm	Food sterilisation, atomic structures
X-rays	1pm to 1nm	Diagnosis, radiotherapy, astronomy
Gamma rays	1fm to 1pm	Diagnosis, radiotherapy

All electromagnetic waves travel in vacuum with the same speed of $c = 3.0 \times 10^8$ metres per second.

All electromagnetic waves transfer energy from one place to another and can be absorbed and emitted by matter.

11.1.3 Wave Characteristics

Definition 11.6: Displacement (Waves)

Displacement is the distance in a specific direction of a point on the wave from its equilibrium position.

Definition 11.7: Amplitude (Waves)

The **amplitude** is the maximum displacement of any point on the wave from its equilibrium position.

Definition 11.8: Period (Waves)

The **period** T is the time taken for one complete oscillation of a point in a wave.

Definition 11.9: Frequency (Waves)

The **frequency** f is the number of oscillations per unit time of a point on a wave.

$$f = \frac{1}{T}$$

.

Definition 11.10: Wavelength

The **wavelength** λ is the **minimum distance** between any two points of the wave with the **same phase** at the same instant.

Definition 11.11: Wave Speed

The **wave speed** v is the speed at which energy is transmitted by a wave.

$$v = \frac{\lambda}{T} = f\lambda$$

Definition 11.12: Wavefront

A **wavefront** is an imaginary line joining points which are in phase.

- Wavefronts are usually drawn one wavelength apart and represent wave crests.
- They are always perpendicular to the direction of wave propagation.
- All the points on a wavefront have the same distance from the source of the wave.

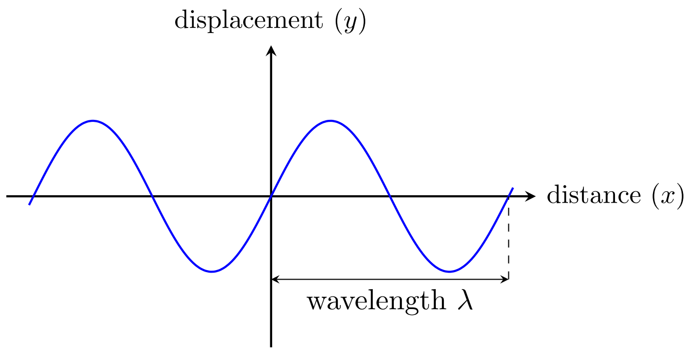
Definition 11.13: Ray

A **ray** is a line pointing in the direction of energy transfer. Rays are always perpendicular to wavefronts.

11.1.4 Graphical Representations

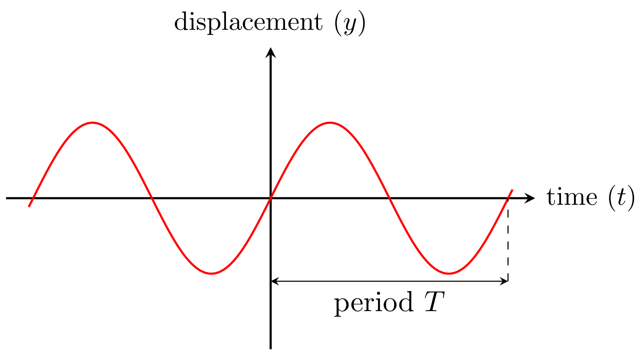
Displacement-Distance Graph

The displacement-distance graph shows how particle displacement varies with the distance from the source at any chronological instant. The **wavelength** can be deduced from the graph.



Displacement-Time Graph

The displacement-time graph shows how the displacement of a singualr wave particle varies with time. The **period** can be deduced from the graph.



11.1.5 Phase

Definition 11.14: phase

The **phase** ϕ is an angle which gives a measure of the **fraction of a cycle** that has been completed by an oscillating particle or wave.

One cycle corresponds to 2π radians.

The **phase difference** $\Delta\phi$ is a measure of how much one wave is out of step with another. Two particles are **in phase** if $\Delta\phi = 0$. Two particles are in **anti-phase** if they are out of phase by a half cycle, where $\Delta\phi = \pi$.

$$\Delta\phi = \frac{\Delta x}{\lambda} \times 2\pi = \frac{\Delta t}{T} \times 2\pi$$

11.2 Cathode Ray Oscilloscope

A **cathode ray oscilloscope** is a device used to study waveforms, measure voltage, and measure short time intervals.

This is usually employed in practicals.

11.2.1 Frequency Determination

In order to determine the *frqeuency* of a sound using a calibrated cathode ray oscilloscope, the following steps are taken.

1. A signal is fed through a microphone into a cathode-ray oscilloscope.
2. With the time-base of the oscilloscope turned on, the trace on the oscilloscope screen will be a display of the displacement against time.
3. The time-base of the oscilloscope is adjusted until a stationary trace is obtained.
4. The period T of the sound wave is given by the product of the length l of the one complete cycle, and the time-base setting b of the oscilloscope.

$$T = lb \implies f = \frac{1}{lb}$$

11.2.2 Wavelength Determination

In order to determine the *wavelength* of a sound using a calibrated cathode ray oscilloscope, the following steps are taken.

1. A loudspeaker delivers a sound through a signal generator; the incident wave is directed towards and reflected at the reflector plate.
2. **Superposition** of the incident wave and the reflected wave (which have equal properties, except opposing directions of travel) produces a **stationary wave** between the loudspeaker and the reflector.
   - This results in a wave with maximum displacement at antinodes, and zero amplitude at nodes.
3. A microphone connected to the oscilloscope is moved along the straight line between the loudspeaker and the reflector.
4. At nodes, minimal signal is displayed, while at antinodes, maximum signal is displayed — the average distance d between two adjacent antinodes can be determined, and thus the wavelength.

$$\lambda = 2d \implies v = f(2d)$$

11.3 Intensity

From simple harmonic motion, we understand that the total energy associated with an oscillatory motion is proportional to the square of its amplitude.

$$E = \frac{1}{2}m\omega^2 {x_0}^2 = \frac{1}{2}m(2\pi f)^2 {x_0}^2$$

Definition 11.15:

The **intensity** I of a wave is the **rate of energy transmitted per unit area** perpendicular to the wave velocity.

$$I = \frac{E/t}{A} = \frac{W}{A}$$

We note that for amplitude x_0 and distance travelled by the wave r,

$$I \propto {x_0}^2 \quad \text{and} \quad I \propto \frac{1}{r^2}$$

Since $I \propto E$ and $E \propto f^2 {x_0}^2$, then for a constant frequency f,

$$I \propto {x_0}^2$$

11.3.1 Inverse Square Law

Consider a point source which emits energy with power P. Energy is transferred radially outward in three dimensions, carried on an expanding spherical surface.

At a distance r from the source,

$$I = \frac{P}{A} = \frac{P}{4\pi r^2} \implies I \propto \frac{1}{r^2} \text{ (3D)}$$

This is known as the **inverse square law**, where the observed intensity of the wave is inversely proportional to the square of the distance from the source.

Notably, the inverse square law **does not hold for two-dimensional transmission**. For a point source that emits energy two-dimensionally, the wave propagation can be encapsulated within a cylinder of height $h = 2x_0$. At a distance r from the source,

$$I = \frac{P}{A} = \frac{P}{2\pi r h} \implies I \propto \frac{1}{r} \text{ (2D)}$$

Example 11.1

A constant power source of 500 watts emits light uniformly in all directions.

1. Calculate the intensity of the light detected by a man located 3.0m from the source.
2. The power of the source is now halved. In order to detect the same intensity as before, determine the distance of the man from the source.

For part (a),

$$I = \frac{P}{4\pi r^2} = \frac{500}{4\pi (3.0)^2} = \boxed{4.42 \text{ W m}^{-2}}$$

For part (b),

$$\frac{P}{4\pi r^2} = \frac{P'}{4\pi R^2}$$

$$R^2 = \frac{P'}{P} r^2 = \frac{1}{2}(3.0)^2$$

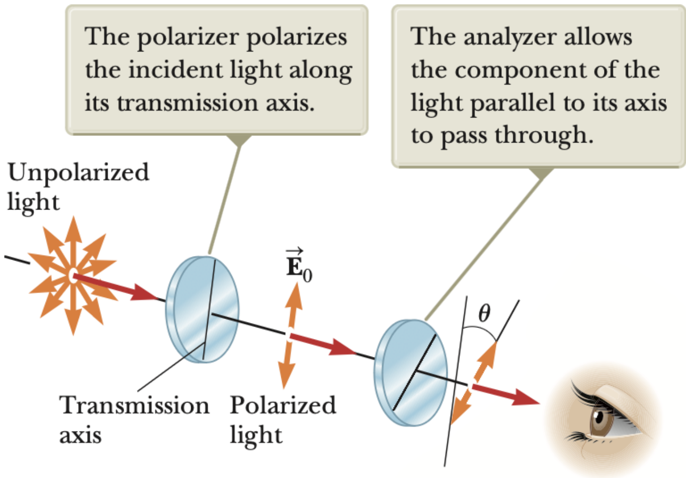
$$R = \boxed{2.12 \text{ m}}$$

11.4 Polarisation

Definition 11.16: Plane Polarisation

If the oscillations in a **transverse wave** are confined to **one direction only**, in a plane normal to the direction of energy transfer, the wave is said to be **polarised** in that direction.

An **ideal polariser** passes all incident light parallel to its polarising axis, and blocks all light perpendicular to this axis.



Since the vertical and horizontal components of random incident light are equal on average, the intensity I of emerging polarised light from an unpolarised light of intensity I_0 is

$$I = \frac{1}{2} I_0$$

It is **not possible to polarise sound waves**, as they are longitudinal, having no oscillations in the plane normal to the direction of transfer of energy of the wave.

11.4.1 Malus's Law

Definition 11.17: Malus's Law

Malus's law states that when an ideal polariser is placed in a polarised beam of light, the intensity I of the light that passes through is

$$I = I_0 \cos^2 \theta$$

where I_0 is the initial intensity of the light and θ is the angle between the light's initial polarisation direction and the axis of the polariser.

This is because when two polarisers are placed along axes varied by an angle θ, only the parallel component of amplitude $A \cos \theta$ is transmitted past the second polariser.

$$\begin{aligned} I_0 \propto A^2,\ I \propto (A\cos\theta)^2 &\implies \frac{I}{I_0} = \cos^2\theta \\ &\implies I = I_0 \cos^2 \theta \end{aligned}$$

12 Superposition

12.1 Superposition Principle

Definition 12.1: Superposition Principle

The **superposition principle** states that when two or more waves of the same type meet at a point at the same time, the **displacement** of the resultant wave is the **vector sum** of the displacements of the individual waves at that point in time.

Two waves must be of the same type in order for the principle of superposition to apply.

12.2 Interference

12.2.1 Coherence

Definition 12.2: Coherence

Two sources are said to be **coherent** if the waves produced by the two sources have a **constant phase difference**.

The satisfaction of coherence implies the propagation of an identical frequency.

12.2.2 Constructive and Destructive Interference

Definition 12.3: Interference

Interference occurs when two or more wvaes overlap to produce a resultant wave pattern.

Constructive interference occurs when two waves **meet in phase**; the resultant amplitude is the sum of the individual amplitudes of the waves.

Destructive interference occurs when two waves **meet in anti-phase**; the resultant amplitude is zero.

The distance along any path from source to receiver is called the **path length** r. The **path difference** between two waves is simply the absolute difference between their path lengths.

Constructive interference occurs when

- the path difference $\Delta r = n\lambda,\ n \in \mathbb{Z}$; or
- the phase difference $\phi = 2k\pi,\ k \in \mathbb{Z}$

Conversely, **destructive interference** occurs when

- the path difference $\Delta r = (n + \frac{1}{2})\lambda,\ n \in \mathbb{Z}$; or
- the phase difference $\phi = (2k + 1)\pi,\ k \in \mathbb{Z}$

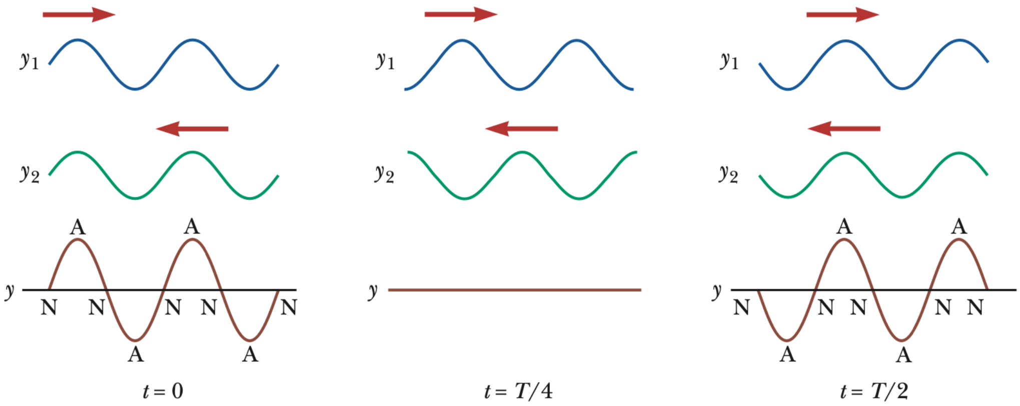
The above is true if both waves are **initially in phase**. If the two waves start in anti-phase instead with $\phi = \pi/2$, the above formulae will **swap** between constructive and destructive interference.

12.3 Stationary Waves

Definition 12.4: Stationary Wave

A **stationary wave** (or **standing wave**) is formed when two progressive waves of the same type of equal amplitude, equal frequency, equal speed ravelling in opposite directions meet and undergo superposition with each other.

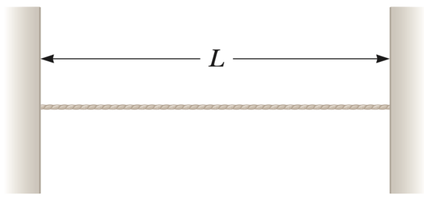
Note that vibrational energy is not transmitted across the wave (hence the name "stationary"). Every stationary wave has amplitudes of oscillation varying from maximum at **antinodes** to zero amplitude at **nodes**.



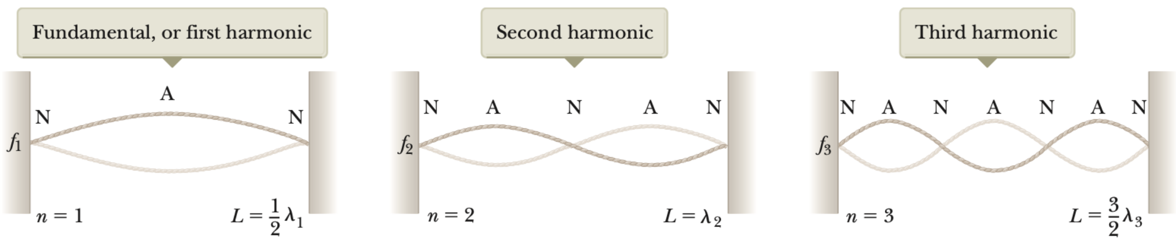
The distance between adjacent antinodes/nodes is $\lambda/2$, whereas the distance between a node and an adjacent antinode is $\lambda/4$.

12.3.1 Two Fixed Ends

Waves in stretched strings contain the following circumstance — the ends of a string are fixed with zero displacement.



This results in a number of discrete oscillatory patterns, called **normal modes**. These modes are also known as **harmonics**.

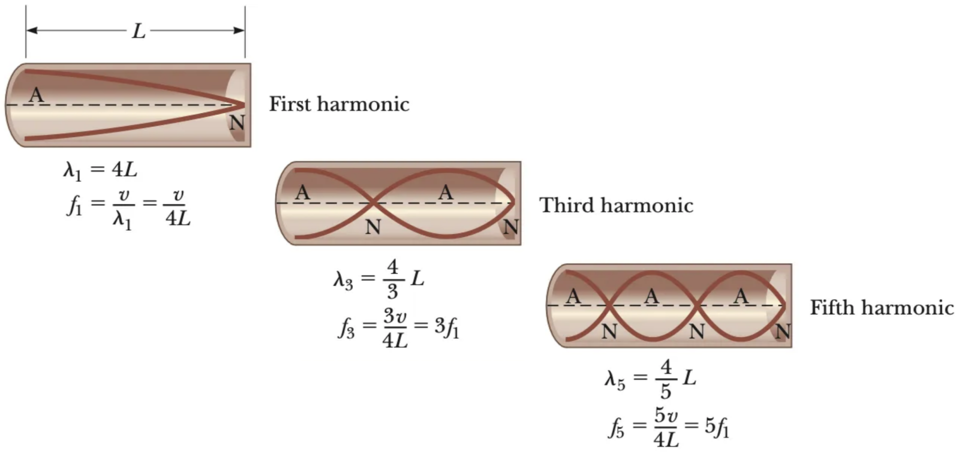


Suppose a standing wave is formed over a stretched string of length L. Then, for the nth normal mode, the wavelength and frequency of the wave is given by

$$\lambda_n = \frac{2L}{n} \implies f_n = \frac{nv}{2L} = nf_1$$

12.3.2 1 Fixed End, 1 Free End

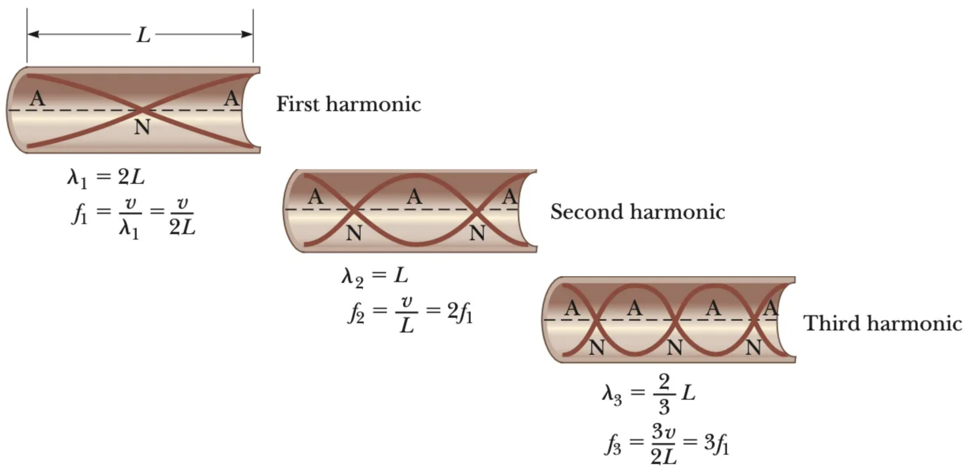
We often examine waves under boundary conditions in air columns — a closed end of a pipe is a **displacement node**. Recall that pressure is 90° out of phase with displacement, hence it is also a *pressure antinode*. This is similar for the open end of a pipe.



In a pipe **closed at one end**, the natural frequencies form a harmonic series that includes *only odd integer multiples* of the fundamental frequency. For a pipe of length L,

$$\lambda_n = \frac{4L}{n} \implies f_n = \frac{nv}{4L}$$

12.3.3 2 Free Ends



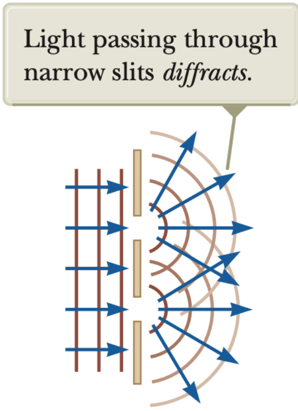
In a pipe **open at both ends**, the natural frequencies form a harmonic series that includes *all integer multiples* of the fundamental frequency.

$$\lambda_n = \frac{2L}{n} \implies f_n = \frac{nv}{2L}$$

12.4 Diffraction

Definition 12.5: Diffraction

Diffraction is the spreading of a wave as it passes through a gap or around an obstacle.



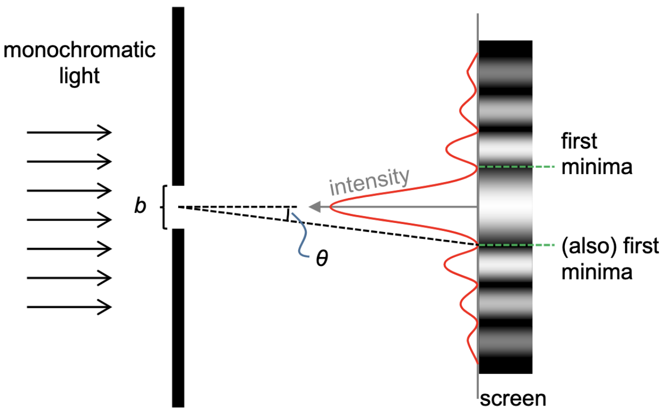
In general, diffraction is most significant when the size of the width is of the same order of magnitude as the wavelength of the wave.

12.4.1 Single-Slit Diffraction

For a **single-slit diffraction** with slit width b, dark fringes occur at angles θ given by the following equation.

$$\sin\theta = m\frac{\lambda}{b},\ \ m \in \mathbb{Z}$$

The figure below shows a typical set-up for single-slit diffraction.



Questions will likely test on what occurs to the diffraction pattern when the fundamental setup is modified. If so, talk about the (1) **separation of fringes**, (2) **amplitude of waves**, and (3) **overlapping of diffracted waves**.

The single slit equation fails with $m\lambda/b > 1$. For such values of m, no diffraction pattern can be observed.

12.4.2 Rayleigh Criterion

Definition 12.6: Rayleigh Criterion

The **Rayleigh criterion** states that two patterns are *just* resolved when the central maximum of one pattern lies on the first minimum of the other pattern.

$$\theta_{\min} \approx \frac{\lambda}{b}$$

where $\theta_{\min}$ is the minimum angle between two rays.

12.4.3 Double-Slit Diffraction

Definition 12.7: Conditions for Double-Source Interference

For double-source interference fringes to be **observable**,

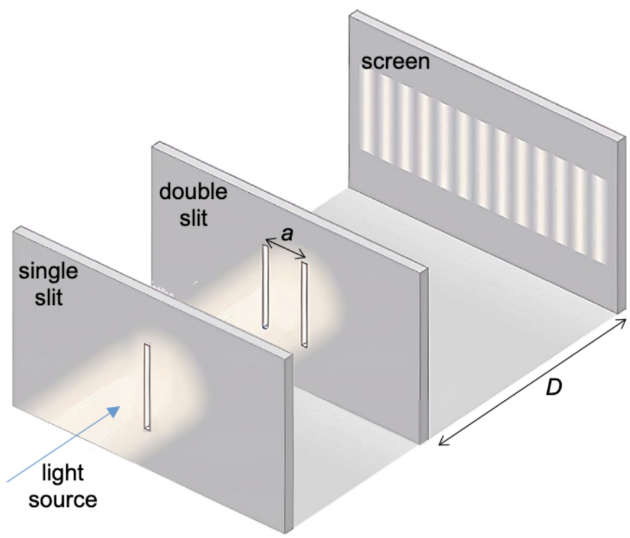
- waves must **meet**
- waves must be **coherent**
- waves have (approximately) **equal amplitudes**
- transverse waves must be either unpolarised or polarised in the same plane
- the slit separation d is of the same order as wavelength λ.

Young's Double-Slit Experiment

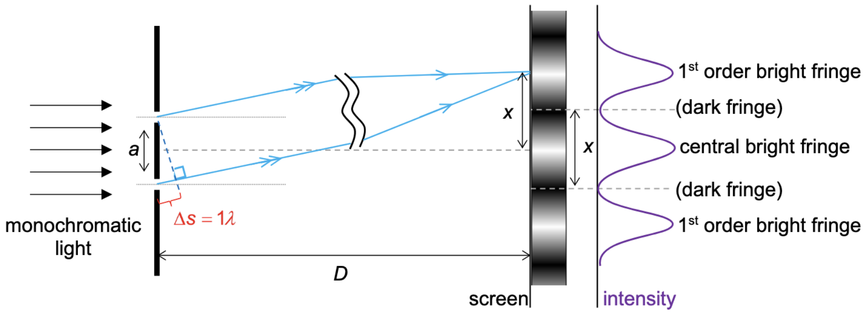
In the case of **Young's double-slit experiment**, for slit separation a, slit-to-screen distance D, and fringe separation x,

$$\lambda = \frac{xa}{D}$$

Young's double-slit experiment consists of a light being passed through a single slit, and then a double slit to form an interference pattern.



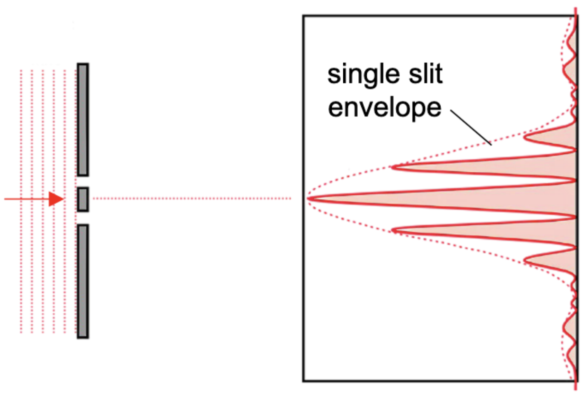
The following diagram shows a visualisation of the wavelength propagation during the double-slit experiment.



Single-Slit Envelope

Since the experiment consists of a double slit in front of a single slit, the primary effects of single-slit diffraction will be observed **over** that of the double-slit interference pattern.

The overall intensity of a double-slit interference pattern is **modulated** by the overarching single-slit envelope.



Without the single-slit envelope, the relative intensities of the bright fringes will instead be constant.

12.4.4 Diffraction Grating

A diffraction grating consists of a large number of equally-spaced lines, each of which are capable of diffracting incident light.

In a diffraction grating with slit separation d, for the nth order maximum, the path difference is given by

$$d \sin\theta = n\lambda$$

Diffraction gratings are usually specified by the number of lines N per unit length.

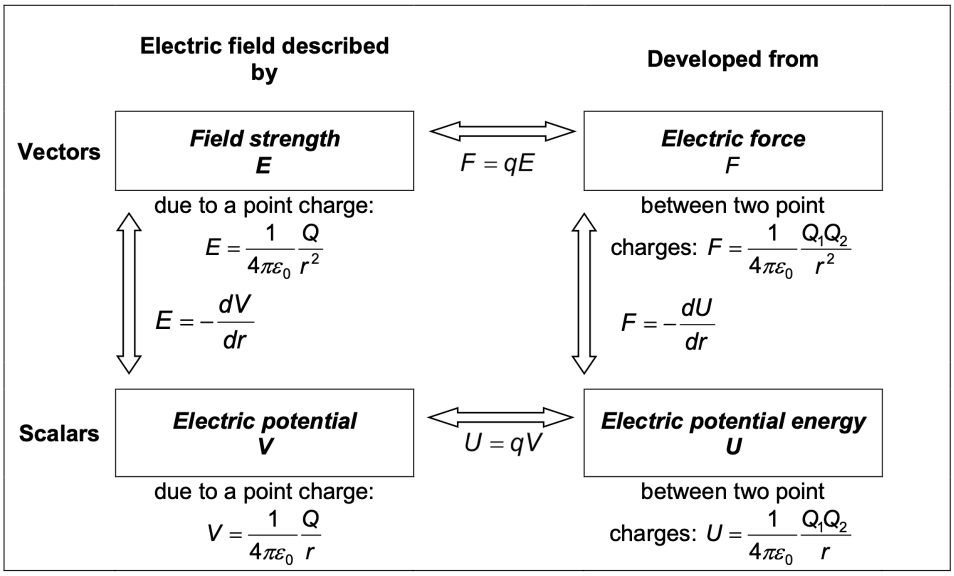
$$d = \frac{\text{unit length}}{N}$$

Part V

Electricity and Magnetism

13 Electric Fields

A map summarising all the relations between different electrostatic quantities is shown below.



13.1 Electric Force

Definition 13.1: Coulomb's Law

Coulomb's law states that the **electric force between two point charges** is **directly proportional to the product of the charges**, and inversely proportional to the **square of the distance** between them.

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

where q_1 and q_2 are the charges (in coulombs, C), r is their separation, and ϵ_0 is the *permittivity of free space*. Note that $1/(4\pi\epsilon_0) = 9.0 \times 10^9$ N m^2 C^{-2}.

The direction of the force is along the line joining the two point charges. This relation is **only applicable** if

- the charges are point charges, or
- if the two charged bodies are separated far apart such that the size of each charged body is negligible compared to their separation.

13.2 Electric Field

Definition 13.2: Electric Field

An **electric field** is a region of space within which there is an electric force acting on a charged object placed in that field.

The direction of the electric field is the direction that an electric force would act upon a positive test charge.

13.2.1 Electric Field Strength

Definition 13.3: Electric Field Strength

The **electric field strength** E at a point in the field is defined as the **electric force per unit positive charge** experienced by a charge placed at that point.

$$E = \frac{F}{q} = \frac{Q}{4\pi\epsilon_o r^2}$$

If a particle of charge q experiences an electric force of F_E when placed at a certain point in the field caused by another charge q, the strength of the electric field caused by charge q at that point is given by the equation above.

Consider a point charge Q. If another point charge q is placed a distance r from Q, the electric force exerted by Q on q is given by Coulomb's law.

$$F = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}$$

Thus, the **electric field strength due to a point charge** is given by

$$E = \frac{F}{q} = \frac{Q}{4\pi\epsilon_o r^2}$$

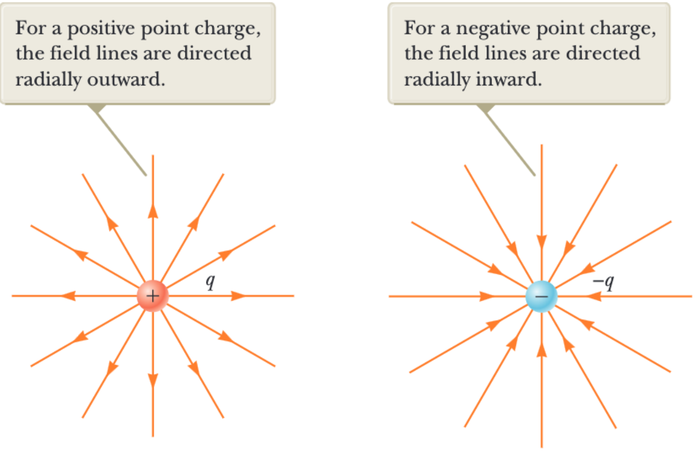
13.2.2 Field Lines

An electric field may be represented by lines of force known as **electric field lines**, which indicate the **direction of the force** a stationary positive charge would experience if placed at that point.

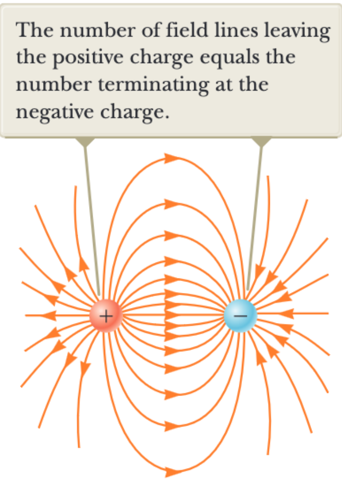
Some basic fields of electric field lines are listed below.

- The **direction** of the electric force is **tangential** to the field line at each point — it is the direction of the force that a small stationary positive test charge would experience.
- The spacing between the lines indicates field strength. A **stronger field** would have **more closely spaced lines**.
- Electric field lines either originate on positive charges or come in from infinity, and either terminate on negative charges or extend out to infinity.
- The number of lines originating from a positive charge or terminating on a negative charge must be proportional to the magnitude of the charge.
- They **do not intersect**.
- In a uniform field, all lines are parallel and equally spaced apart.
- Field lines must touch the surface of charged conductors at a right angle.

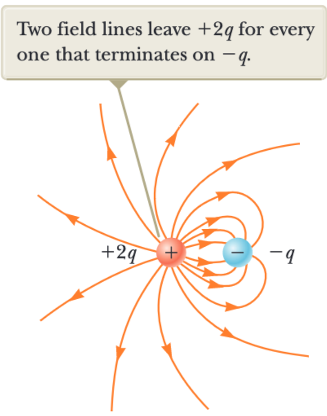
For an isolated positive charge, its field lines point radially outward. Conversely, for an isolated negative charge, its field lines point radially inward.



For two charges of equal magnitude, they cancel out. The number of field lines leaving the positive charge must be equal to the number of field lines terminating at the negative charge.

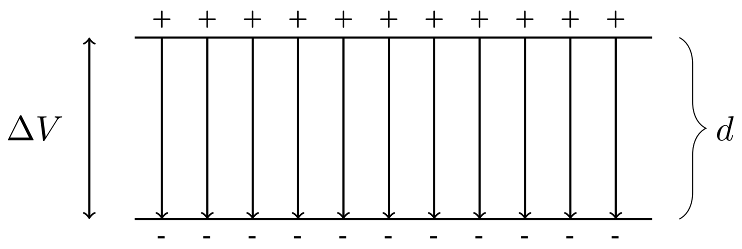


Similarly, for a positive charge $+2q$ and a negative charge $-q$, two field lines must leave the positive charge for every line that terminates on the negative charge



13.2.3 Charged Parallel Plates

Consider a pair of charged parallel plates separated by a distance d with potential difference ΔV, as shown below.



The **uniform electric field E between both plates** has a field strength with magnitude

$$E = \frac{|\Delta V|}{d} \implies F = \frac{q|\Delta V|}{d}$$

Note that the direction of the electric field points from the plate with the higher electric potential to the plate with the lower electric potential.

13.2.4 Motion of Charged Particles

Within a uniform electric field, the force on a charged particle anywhere in the field is **constant**, meaning that the charged particle has **constant acceleration**.

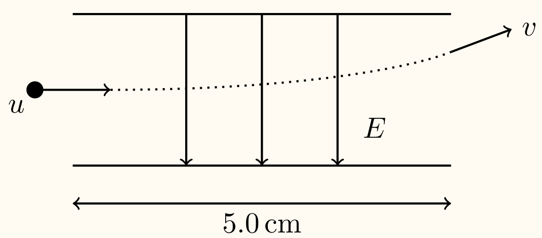
Depending on the relative direction of electric field to its motion, it could be in either linear motion or projectile motion. In both cases, under the electric force, a charged particle of mass m will accelerate with an acceleration a.

$$\sum F = ma \implies qE = ma \implies \frac{q|\Delta V|}{d} = ma \implies a = \frac{q|\Delta V|}{md}$$

As the charged particle has constant acceleration, its kinematic quantities can be modelled using the **kinematic equations of motion**. For t, the equation $s = ut$ is commonly used along an axis where the charged particle does not experience any acceleration.

Example 13.1

An electron is projected into the space between two charged plates of length 5.0 cm, as shown in the diagram below.



The electric field strength E is 3.5×10^4 N C^{-1} and the velocity u of the electron is 3.2×10^7 m s^{-1}. Neglecting gravitational effects, determine

1. the distance it deviates towards the positively-charged plate, and;
2. its speed just as it leaves the electric field.

To determine vertical deviation, we observe that the electron experiences no horizontal acceleration. Hence,

$$s_x = u_x t \implies t = \frac{0.05}{3.2 \times 10^7} = 1.56 \times 10^{-9} \text{ s}$$

From Newton's second law,

$$F = ma = qE \implies a = \frac{qE}{m}$$
$$\implies a = \frac{(1.60 \times 10^{-19})(3.5 \times 10^4)}{9.11 \times 10^{-31}} = 6.15 \times 10^{15} \text{ m s}^{-2}$$

Using kinematic equations,

$$\begin{aligned} s_y &= u_y t + \frac{1}{2} a_y t^2 \\ &= 0 + \frac{1}{2}(6.15 \times 10^{15})^2 (1.56 \times 10{-}9)^2 \end{aligned}$$

This gives us the vertical deviation of the electron.

$$\boxed{s_y = 7.48 \times 10^{-3} \text{ m}}$$

To find the final velocity of the electron, we note that $v_x = 3.2 \times 10^7 \text{ m s}^{-1}$. Finding the velocity's vertical component,

$$\begin{aligned} v_y = u_y + a_y t &= 0 + (6.15 \times 10^{15})(1.56 \times 10^{-9}) \\ &= 9.59 \times 10^6 \text{ m s}^{-1} \end{aligned}$$

The velocity is hence

$$v = \sqrt{{v_x}^2 + {v_y}^2} = \boxed{3.34 \times 10^7 \text{ m s}^{-1}}$$

13.3 Electric Potential

Definition 13.4: Electric Potential

The electrical potential, V, at a point is the **work done per unit positive charge** by an external agent to bring a small test charge q from infinity to that point, without producing any acceleration.

$$V = \frac{W}{q} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

where V is the electric potential at a distance r from the centre of Q.

Electrical potential is a *scalar* quantity with S.I. unit volt (V) or joule per coulomb.

13.3.1 Equipotential Lines

Electric potential in an electric field can be represented by **equipotential lines** indicated by a dashed line. Equipotential lines are lines that join points in an electric field with the **same electric**

potential. They meet **perpendicular** to the electric field lines.

No work is done by an external force when a charge is moved **along** an equipotential line, because the electric force is always perpendicular to said line. Work is only done if the charge has some component of its motion parallel to the electric force.

13.3.2 Electric Potential Gradient

Definition 13.5: Electric Potential Gradient

The **electric potential gradient** at a point in an electric field is **numerically equal** to the electric field strength E.

$$E = -\frac{dV}{dr}$$

The term dV/dr is known as the potential gradient. Hence, the field strength at a point in the field is the negative derivative of electric potential V with respect to r. The negative sign indicates the direction of the electric field, which **points towards the decreasing electric potential**.

13.4 Electric Potential Energy

Definition 13.6: Electric Potential Energy

When two point charges q_1 and q_2 have a separation r, the **electric potential energy** U of the system of two charges is given by

$$U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r} = qV$$

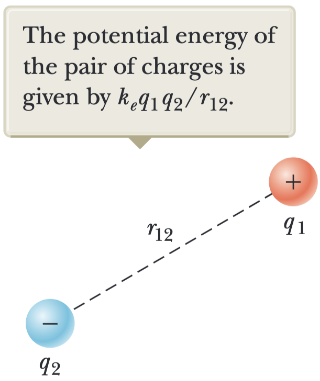
Where q_1 and q_2 are **like charges**,

- U is **always positive**.
- As $r \to 0$, $U \to \infty$. As $r \to \infty$, $U \to 0$.
- U increases as they come closer, since work is done on the repelling system.

Where q_1 and q_2 are **unlike charges**,

- U is **always negative**.
- As $r \to 0$, $U \to -\infty$. As $r \to -\infty$, $U \to 0$.
- U decreases as they come closer, since work is done by the attracting system.

U is a shared property between two charges q_1 and q_2, arising as a consequence of the interaction between these two bodies.



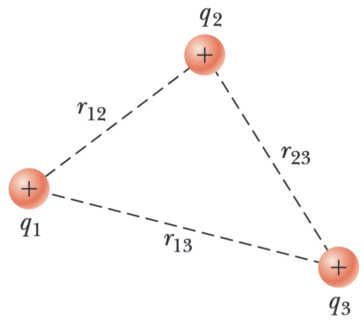
For a **system of several charges** containing $q_1, q_2, \cdots, q_n$, the total electric field at each point is the vector sum of the fields due to the individual charges, and the *total work done on a single point charge* Q during any displacement is the sum of the contributions from the individual charges.

$$U = \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots + \frac{q_n}{r_n}\right)$$

As for a system of several charges, we can find the *total electric potential energy of the system* by calculating U for every *pair* of charges. Expressed in sigma notation, the generalised equation for the total electric potential energy of a system is

$$\sum U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}$$

In particular, the figure below depicts a system of three point charges.



Enumerating through all pairs of charges, the total electric potential energy of this system is

$$\sum U = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_3}{r_{13}}\right)$$

14 Current of Electricity

Without electricity, there can be no art.

— Nam June Paik

14.1 Charge

Charge is a property possessed by some elementary particles, most notably electrons and protons. Electrical charge (commonly represented by Q) has a S.I. unit of coulomb (C).

Charge comes in two varieties — **positive and negative**, which neutralise one another if brought together in equal quantities. Objects that have an electrical charge exert electrostatic forces on each other.

The **elementary charge**, $e = 1.60 \times 10^{-19}$C, is the charge of one proton.

14.2 Electric Current

Definition 14.1: Electric Current

Electric current is the **rate of flow of charge**. It has the S.I. unit of ampere (A).

$$I = \frac{dQ}{dt} \implies \langle I \rangle = \frac{Q}{t}$$

Electric current consists of *charges in motion* from one region to another. When this motion takes place within a conducting path in a closed loop, the path is an **electric circuit**.

In metallic conductors, current is caused by the flow of electrons. However, by convention, the direction of current is the direction of positive charges' movement.

14.2.1 Microscopic Model of Current

Definition 14.2:

Current is the rate of flow of charge, and is thus the **number of charge carriers** passing through **per unit time** multiplied by the **charge per carrier**.

$$I = nAvq$$

where n is the charge carrier density, A is the cross-sectional area of the conductor, v is the drift velocity, and q is the charge per carrier.

Collisions between electrons and metal ions caused by thermal energy are frequent and random, but thermal motion itself does not carry the electron any distance along the wire.

Rather, due to the **potential difference** applied across the conductor, the charge carriers *gain an additional drift motion* that carries them along the conductor ($\approx 10^{-4}$ m s^{-1}), causing charge to be transferred. The direction of this drift is determined by field lines from the cell.

14.3 Voltage

14.3.1 Potential Difference

Definition 14.3: Potential Difference

The **potential difference** between **two points** in a circuit is defined as the work done per unit charge when *electrical energy is converted to non-electrical energy* when the charge passes from one point to the other.

$$\text{potential difference} = \frac{W}{Q}$$

where W is the energy transformed from electrical to non-electrical energy when a charge of Q flows in the part of the circuit.

When a current passes through an electrical device, electrical energy is converted into other forms of energy (e.g. heat, light, mechanical energy).

The amount of electrical energy converted per unit charge is called the **electrical potential difference**.

14.3.2 Electromotive Force

Definition 14.4: Electromotive Force

The **electromotive force** (abbreviated **e.m.f.**) of a source can be defined as the work done per unit charge when *non-electrical energy is converted into electrical energy* when the charge is moved around a complete circuit.

In order to maintain an electric current in an electrical circuit, a potential difference across the ends of the entire circuit is needed. To achieve this, the ends of the electrical circuit are connected to opposite terminals of a battery.

This potential difference maintained by the battery is known as its **electromotive force**, a measure of work done per unit charge by the battery.

Electromotive force	**Potential difference**
Refers only to the source.	Refers to any two points in an electrical circuit.
Considers the amount of non-electrical energy converted into electrical energy per unit charge passing through the source.	Considers the amount of electrical energy converted to non-electrical energy per unit charge passing from one point to another.
It is a source of energy that *exists irregardless of whether a current is flowing.*	It can only exist if current is flowing in the electrical current.

14.4 Resistance

Definition 14.5: Resistance

The **resistance** of a component is defined as the ratio of the potential difference across the component to the current passing through it.

$$R = \frac{V}{I}$$

The resistance of a conductor is a measure of its opposition of flow of current through itself — it is *a property of a conductor*.

14.4.1 Ohm's Law

Definition 14.6: Ohm's Law

Ohm's Law states that the current flowing through a conductor is **directly proportional** to the potential difference applied across the conductor, **provided that physical conditions remain constant**.

$$I \propto V$$

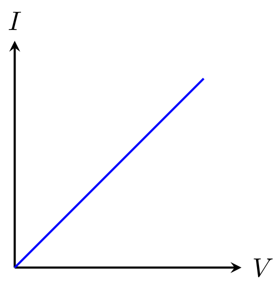
14.4.2 I-V Characteristics

The I-V graph of a material, obtained experimentally, allows the resistance of the material to be determined for various values of I or V.

When the current passes through a material increases, the **temperature** of the material is likely to **increase**. When the latter occurs, two main changes occur.

1. The **charge concentration** n **increases**, which **reduces** the resistance of the material.
2. The **thermal vibrations of the lattice atoms** increases, which **increases** the resistance of the material.

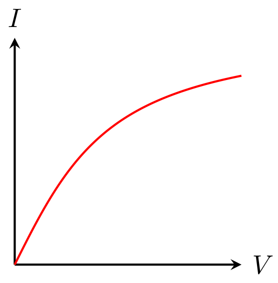
Metallic Conductor



I/V graph of a metallic conductor at constant temperature.

Since the conductor is maintained at constant temperature, the charge concentration n is **fixed** and the amplitude of atomic vibration and the rate of collisions with the atoms **remains unchanged**.

Thus, the resistance remains a constant and the conductor is said to be **ohmic**.



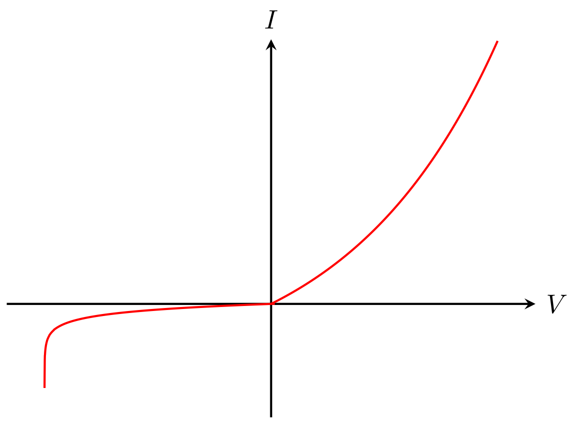
I/V graph of a filament lamp.

Filament Lamp

As V increases, I increases initially. Further increase of V causes a less-than-proportionate increase in the current.

As the temperature increases, n will *not* vary significantly, but the **amplitude of atomic vibrations increases**, and so does the number of collisions with the atoms per unit time. Since the increase in the rate of interaction of electrons with the lattices **predominates over** the increase of n, the overall effect is that R increases.

Semiconductor Diode



I/V graph of a semiconductor diode.

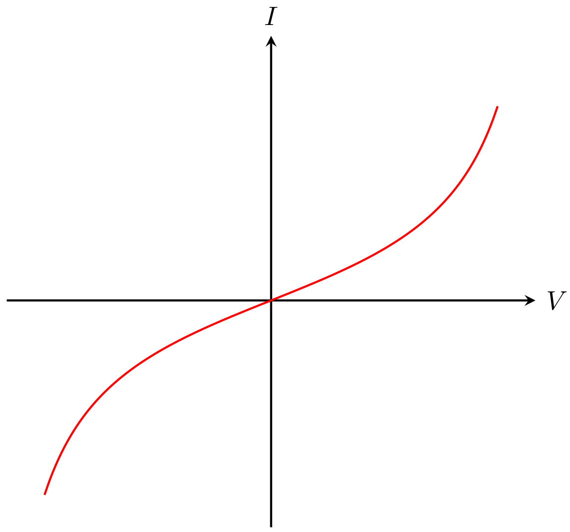
Considering the **forward-biased** region,

- As V increases, the temperature of the semiconductor increases.
- Electrons in the semiconductor are more likely to have sufficient energy to escape from a particular atom if the temperature is higher, hence n **increases significantly**.
- The rate of interaction of electrons with the vibrating atoms **also increases**.
- However, as the increase in n **predominates over** the rate of interactions of electrons with the lattice, the overall effect is that R **decreases**.

If the forward voltage is greater than a threshold value (0.7V for silicon diodes), then a large current will pass through the diode.

For the **reverse-biased region**, there is no current flow through the diode until the breakdown voltage (at least 50V).

NTC Thermistor



I/V graph of a NTC thermistor.

For a **negative temperature coefficient** (abbreviated **NTC**) thermistor, its resistance **falls** at the temperature **rises** (i.e. its resistance is inversely proportional to temperature).

As the temperature increases, the charge concentration n of the material increases, hence reducing the resistance. As such, its I/V characteristic is the same as that for a semiconductor diode for positive values of V and I.

14.4.3 Resistivity

Definition 14.7: Resistivity

For a conductor of resistivity ρ, length l, and cross-sectional area A, its resistance is given by

$$R = \rho \frac{l}{A}$$

The resistance of a conductor depends on

- the nature of the material;
- the size of the sample (length and cross-sectional area); and
- the temperature of the sample.

Consider the analogy of water flow — a longer pipe offers more resistance, while a wider pipe offers less resistance than a narrow pipe. Hence, one would expect electrical resistance to be greater with a longer and thinner conductor.

The constant of proportionality ρ, known as **resistivity** (dependent on temperature), is *governed by the nature of the material.* Metals generally have low resistivity values ($\approx 10^{-8}\,\Omega\,\mathrm{m}$), whereas insulators have high resistivity values ($10^{13} - 10^{16}\,\Omega\,\mathrm{m}$).

14.5 Electrical Power

Definition 14.8: Electrical Power

Electrical power, denoted by P is electrical energy dissipated by a conductor per unit time. **Watt's law** provides the following relationships between power, voltage, and current.

$$P = VI = I^2R = \frac{V^2}{R}$$

.

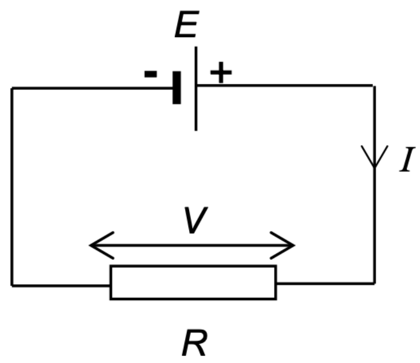
14.5.1 Power Rating

The rated power is the rate at which energy is used (power consumption) by a device when the device is operating at the rated potential difference across it.

For example, when an electrical device is rated 100W and 220V, it means that when a potential difference of 220V is applied across the terminals of the device, the device dissipates 100W of power.

14.6 Internal Resistance

Consider a simple circuit consisting of a cell and a resistor.



In an **ideal circuit**, the connecting wires would have no resistance, and the potential difference V across the cell terminals is **equal** to the electromotive force E of the cell.

However, **real** cells always have some internal resistance r — not all electrical energy generated is available to the external load R.

Some energy is lost as heat within the source due to its internal resistance, hence the **terminal p.d.** is **not equal** to the e.m.f. of the source.

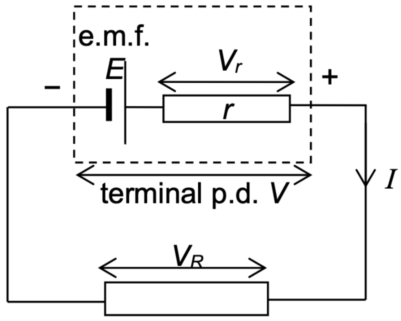
A **real battery** can be modelled as (one ideal cell or a combination of ideal cells) connected in series to a resistor r.

The power supplied by the ideal source is given by

$$P_s = IE$$

In the internal resistance r, there is power dissipated as heat.

$$P_r = I^2r = IV_r \text{ (dissipated as heat)}$$



In the external load R,

$$P_R = I^2 R = IV_R$$

By the principle of conservation of energy,

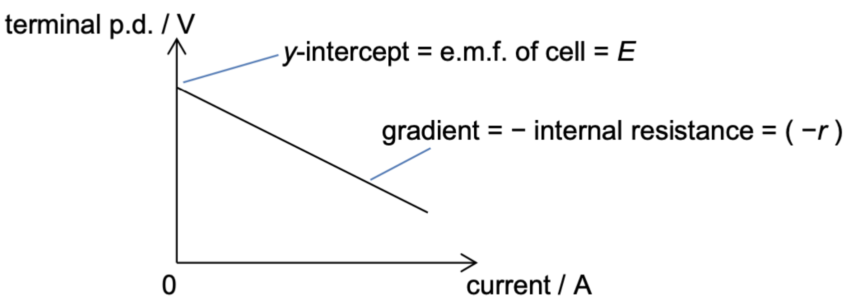
$$\begin{aligned} P_s &= P_r + P_R \\ IE &= I^2 r + I^2 R \\ &\implies \boxed{E = I(r + R)} \end{aligned}$$

Alternatively,

$$\begin{aligned} P_s &= P_r + P_R \\ IE &= IV_r + IV_R \\ E &= Ir + V_R \\ &\implies \boxed{V_R = E - Ir} \end{aligned}$$

Here, V_R is equal to the terminal p.d. across the circuit.

When a current I is flowing, the voltage drops by Ir across the internal resistance and hence the remaining voltage across the battery will be $V = E - Ir = V_R$.



14.6.1 Maximum Power Theorem

Definition 14.9: Maximum Power Theorem

The **maximum power theorem** states that maximum power is supplied to the external circuit when the **resistance** of the external circuit is **equal** to the **internal resistance** of the cell.

Proof. Consider a practical source with e.m.f. E and internal resistance r connected in series with an electrical device of resistance R.

The current passing through the circuit is given by

$$I = \frac{V}{R} = \frac{E}{R+r}$$

Thus, the power delivered to the device is

$$P = I^2 R = \left(\frac{E}{R+r}\right)^2 R$$

To find maximum P, we let the derivative be zero.

$$\begin{aligned}
\frac{dP}{dR} = 0 &\implies \frac{E^2(R+r)^2 - 2E^2R(R+r)}{(R+r)^4} = 0 \\
&\implies E^2(R+r)^2 - 2E^2R(R+r) = 0 \\
&\implies R + r = 2R \\
&\implies \boxed{R = r}
\end{aligned}$$

□

Hence, maximum power is dissipated through the circuit when the resistance of the external circuit is equal to the internal resistance of the cell.

When operating an electrical device at maximum power, the output efficiency of the source is

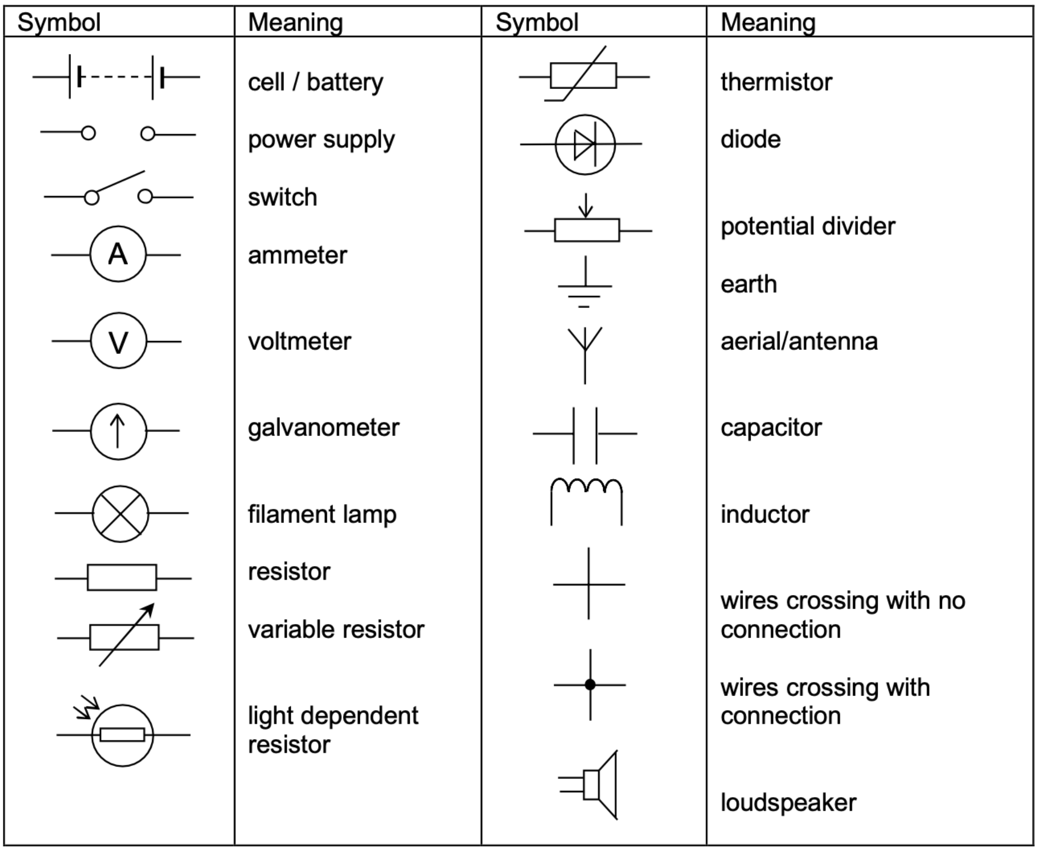
$$\eta = \frac{\text{useful energy}}{\text{total energy}} = \frac{I^2R}{I^2(R+r)} = \frac{R}{R+r} = \frac{R}{2R} = 50\%$$

Half the amount of power it generates is lost due to the internal resistance of the cell.

15 D.C. Circuits

15.1 Symbols

The following image shows some common circuit symbols as specified by the Association of Scientific Education.



Symbol	Meaning	Symbol	Meaning
	cell / battery		thermistor
	power supply		diode
	switch		potential divider
	ammeter		earth
	voltmeter		aerial/antenna
	galvanometer		capacitor
	filament lamp		inductor
	resistor		wires crossing with no connection
	variable resistor		wires crossing with connection
	light dependent resistor		loudspeaker

Ideal voltmeters have infinite internal resistance and zero current, but in practice, it draws a small current and thus reads a potential difference smaller than the actual value. Voltmeters are always connected **in parallel** across the component to be measured.

Ideal ammeters have zero internal resistance and zero potential difference, but in practice, it has a finite potential difference across its ends and thus reads a current smaller than the actual value. Ammeters are always connected **in series** along the network where the current is to be measured.

15.2 Kirchhoff's Laws

Kirchhoff's circuit laws are two equalities that pertain to electrical current and potential difference, and are crucial in the field of circuit analysis.

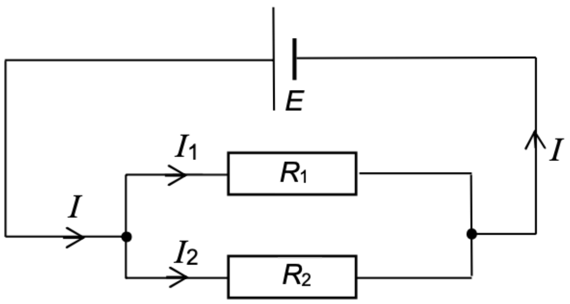
15.2.1 Kirchhoff's Current Law

Definition 15.1: Kirchhoff's Current Law

Kirchhoff's current law states that the sum of currents entering any junction in an electric circuit is always equal to the sum of current leaving that junction.

$$\sum_{i=1}^{n} I_i = 0 \text{ (for a junction)}$$

Kirchhoff's current law is a direct result of the principle of conservation of charge. The total charge that enters a junction per unit time must be equal to the total charge that leaves the same junction per unit time.



In the circuit above, the three currents are related by $I = I_1 + I_2$.

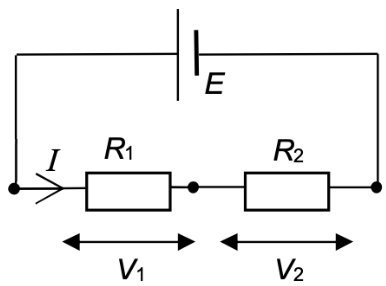
15.2.2 Kirchhoff's Voltage Law

Definition 15.2: Kirchhoff's Voltage Law

Kirchhoff's voltage law states that in any closed loop in an electric circuit, the total electromotive force supplied E equals to the total potential difference in that loop.

$$\sum_{i=1}^{n} V_i = 0 \text{ (for a loop)}$$

The principle of conservation of energy imlpies that the electrical energy produced by the source should be equal to the sum of electrical energy consumed by all the components.



In the circuit above, considering energy,

$$\begin{aligned}
\text{energy from source} &= \text{energy used by resistors} \\
(IE)t &= (I^2R_1)t + (I^2R_2)t \\
E &= IR_1 + IR_2 \\
E &= \boxed{V_1 + V_2}
\end{aligned}$$

15.3 Effective Resistance

For resistors connected in series,

$$R_{\text{eff}} = R_1 + R_2 + \cdots$$

The effective resistance of resistors in series is **always larger** than the **largest resistance** in the circuit.

For resistors connected in parallel,

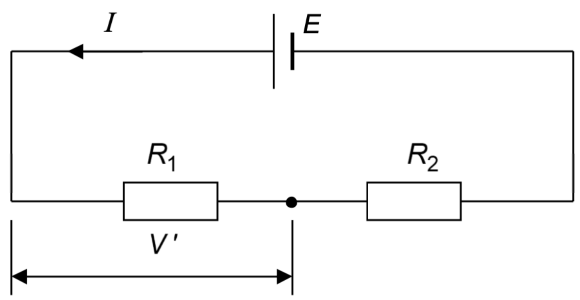
$$R_{\text{eff}} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \cdots\right)^{-1}$$

The effective resistance of resistors in parallel is **always smaller** than the **smallest resistance** in the circuit.

15.4 Potential Divider

A **potential divider** is an arrangement of resistors which is used to obtain a fraction of the potential difference provided by a battery.

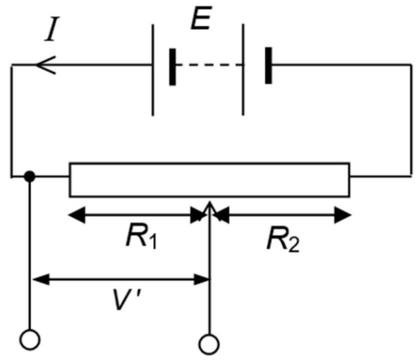
The circuit shows a simple potential divider, consisting of two resistors of resistances R_1 and R_2 connected to a cell with e.m.f. E.



V' here is the potential difference across R_1.

$$V' = IR = \left(\frac{E}{R_{\text{eff}}}\right) R_1 = \left(\frac{E}{R_1 + R_2}\right) R_1$$

To supply a continuously variable potential difference from zero to the full voltage of the supply E, a resistor with a sliding contact may be used, as shown below.



15.4.1 Potential

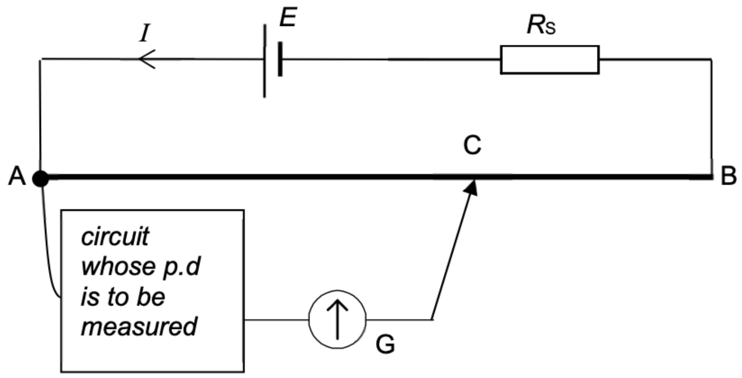
Every point in a circuit has a **potential**.

- Two points have equal potential if they are connected directly by a zero-resistance wire; there is **no** potential difference between them.
- If two points are separated by a component with resistance, they have a **non-zero** potential difference (unless there is no current).

For potential difference to exist, the component must either be an e.m.f. source, or have both resistance and current flowing through it.

15.5 Potentiometer

Consider the setup below.



The circuit consists of a cell with e.m.f. E, a resistor R_s, a resistance wire of length AB. Connected in parallel to the circuit is another circuit with unknown potential difference, a galvanometer, and a sliding contact at C.

The sliding contact will have a point where **there is no current in the galvanometer**. This point is known as the balance point.

Let the balance point be C. The length AC is known as the **balance length**.

The essence is that at the balance point, the **p.d. across AC is equal to the p.d. across the unknown circuit**.

We assume that the wire has uniform cross-sectional area, constant potential difference (with time), and constant length (with time).

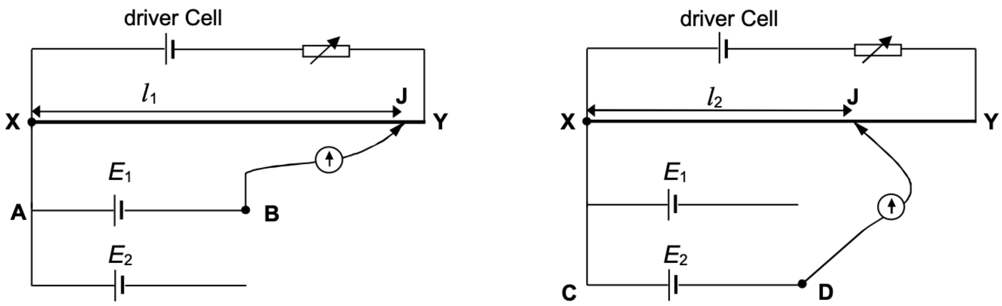
For a uniform wire of length L, $R \propto L$. Since the same current flows through the entire wire,

$$V_{\text{AC}} = \frac{R_{\text{AC}}}{R_{\text{AB}}} \times V_{\text{AB}} = \frac{L_{\text{AC}}}{L_{\text{AB}}} \times V_{\text{AB}} = \frac{V_{\text{AB}}}{L_{\text{AB}}} \times L_{\text{AC}} = \boxed{kL_{\text{AC}}}$$

where k is the **potential per unit length** of the resistance wire AB.

15.5.1 Comparison of Electromotive Force

Suppose you have the setup below. You are trying to find the ratio of E_1 to E_2.



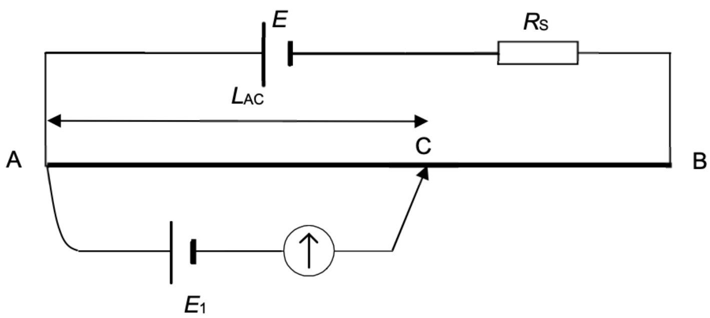
At the balance length,

$$E_1 = kl_1 \text{ and } E_2 = kl_2 \implies \frac{E_1}{E_2} = \frac{l_1}{l_2} \implies \boxed{E_1 = \frac{l_1}{l_2} E_2}$$

The setting of the variable resistor must be the same for the two cells so that the potential difference across wire XY is the same.

15.5.2 Measuring Electromotive Force

Consider the setup below.



A potentiometer has a resistance wire AB of length L and resistance R, powered by a cell of e.m.f. E, alongside a resistor of resistance R_s.

With an unknown e.m.f. E_1 in the branch circuit, the balance point is C.

The p.d. across wire AB is

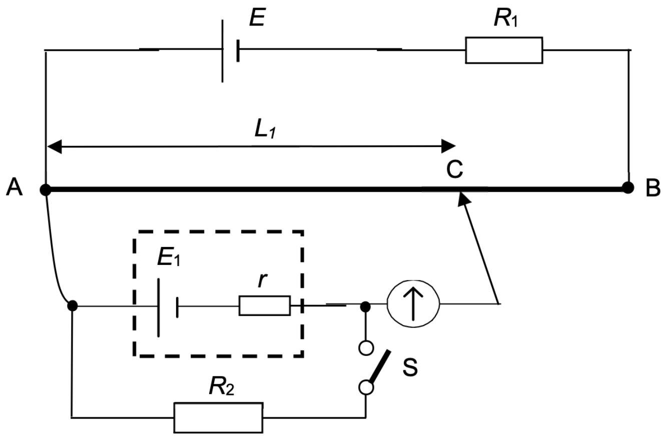
$$V_{\text{AB}} = IR = \frac{R}{R + R_s} \times E = \frac{RE}{R + R_s}$$

Using the potentiometer principle,

$$E_1 = \frac{L_{\text{AC}}}{L_{\text{AB}}} \times V_{\text{AB}} = \boxed{\frac{L_{\text{AC}}}{L}\left(\frac{RE}{R+R_s}\right)}$$

15.5.3 Determining Internal Resistance

To determine the internal resistance of a cell, consider the following circuit.



Switch S is open.

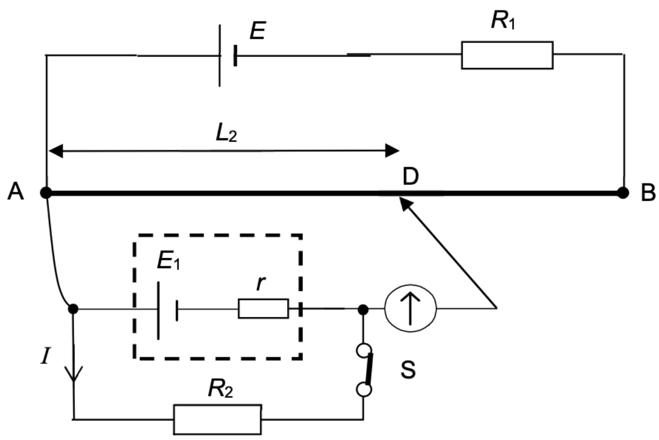
When switch S is **opened**, there is **no current** passing through R_2. The balance point is C with balance length L_1.

From this alone, **because there is no current passing through** r, we can find E_1 with ease.

$$E_1 = \frac{L_{\text{AC}}}{L_{\text{AB}}} \times V_{\text{AB}} = \frac{L_1}{L}\left(\frac{R_{\text{AB}}E}{R_{\text{AB}}+R_1}\right)$$

OK.

Now, we shall close S.



Switch S is closed.

When switch S is **closed**, **current passes through** R_2. The new balance point is D with balance length L_2.

Now, **because there is now current in the bottom circuit**, we can find the **terminal p.d.** V_T across E_1.

$$V_T = \frac{L_2}{L}\left(\frac{R_{\text{AB}}E}{R_{\text{AB}} + R_1}\right)$$

The terminal p.d. V across E_1 will also be equal to the p.d. across resistor R_2. Using $V = IR$, we can find the current flowing through R_2.

$$V = IR \implies I = \frac{V_T}{R_2}$$

Using the equation $V = E - Ir$, we can find the internal resistance r.

$$V = E - Ir \implies r = \frac{E - V_T}{I}$$

16 Electromagnetism

The world may be utterly crazy
And life may be labour in vain;
But I'd rather be silly than lazy,
and would not quit life for its pain.

— James Clerk Maxwell, c. 1858

16.1 Magnetic Fields

Definition 16.1: Magnetic Field

A **magnetic field** is a region of space within which a non-contact magnetic force

- must act on a permanent magnet or ferromagnetic material, and
- *might* act on a current-carrying conductor or a moving charge.

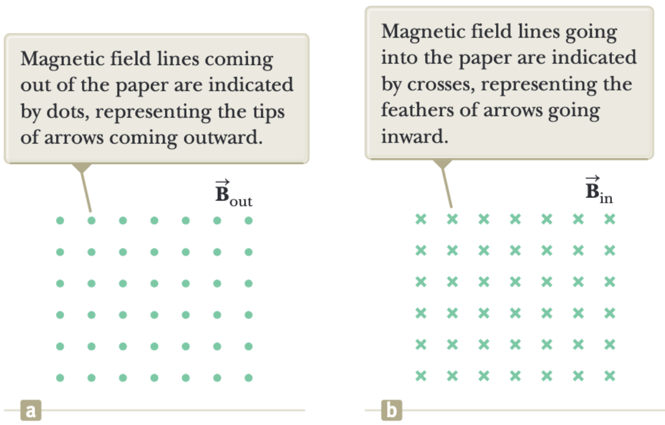
The strength of a magnetic field is expressed by the **magnetic flux density** $\vec{\mathbf{B}}$, a vector quantity with the unit tesla (T).

16.1.1 Magnetic Field Lines

The following are characteristics of magnetic field lines.

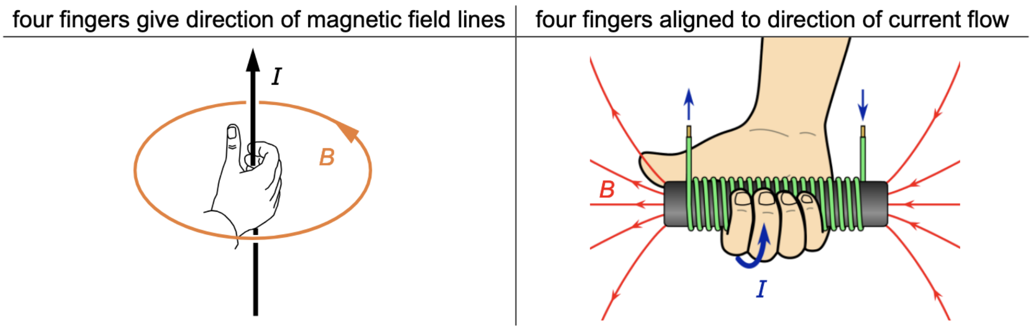
- They come out of north poles and go into south poles.
- The tangent of a field in space shows the direction of force that a "free" magnetic monopole will experience at that point.
- Stronger fields are represented by a denser concentration of lines; conversely, weaker fields are represented by lines that are further apart.

Magnetic field lines that point **into the paper** are denoted by **crosses**, whereas field lines that point **out of the paper** are denoted by **dots**.



16.1.2 Ampere's Right-Hand Grip Rule

Ampere's right-hand grip rule provides a visualisation for the geometrical relationship between magnetic fields and direction of current flow.

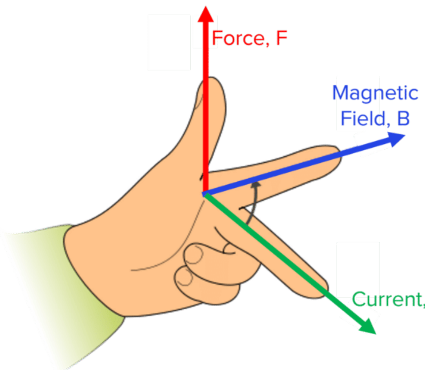


This can be used for straight wires, flat circular coils, and solenoids.

16.1.3 Fleming's Left-Hand Rule

Current-carrying conductors and moving charged particles in a magnetic field might experience a force. This force is **always perpendicular** to the direction of the current and the direction of the magnetic field.

Fleming's left-hand rule provides a handy visualisation for the directions of the magnetic force, magnetic field, and electrical current.



16.1.4 Magnetic Field Intensification

A **ferrous core** can increase the magnetic flux density inside a **long solenoid** compared to an air core solenoid.

This is because the magnetic permeability of the ferrous core ($k\mu_0$) is greater than the magnetic permeability of air (μ_0).

Other methods to increase the magnetic flux density of the long solenoid include

- increasing the current I flowing through the long solenoid, and
- increasing the number of turns per unit length n of the solenoid.

16.2 Magnetic Flux Density

Definition 16.2: Magnetic Flux Density

The **magnetic flux density** B of a magnetic field is defined as the **magnetic force per unit current per unit length** acting on a straight current-carrying conductor placed **perpendicular** to a uniform magnetic field.

$$B = \frac{F}{IL\sin\theta}$$

where B is the magnetic flux density, F is the magnetic force acting on the conductor, I is the current, L is the length of conductor, and θ is the angle between $\vec{\mathbf{B}}$ and $\vec{\mathbf{F}}$.

The S.I. unit of magnetic flux density is the tesla (T). For all following definitions, $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1} is the permeability of free space.

16.2.1 Long Straight Wire

The magnetic flux density of a current-carrying long straight wire is given by

$$B = \frac{\mu_0 I}{2\pi d}$$

where I is the current through wire and d is the perpendicular distance of the point in question from the wire.

16.2.2 Flat Circular Coil

The magnetic flux density of a flat circular coil of radius r with N coil loops, carrying a current I, is given as

$$B = \frac{\mu_0 NI}{2r}$$

16.2.3 Solenoid

The magnetic flux density of an (infinitely) long solenoid carrying a current I is given by

$$B = \mu_0 nI$$

where $n = N/L$ is the number of turns per unit length of the solenoid.

16.3 Magnetic Force

The magnitude of the magnetic force acting on a **current-carrying conductor** placed at an angle θ to the magnetic field can be expressed as

$$\vec{\mathbf{F}} = \vec{\mathbf{B}} \times I\vec{\mathbf{L}} \implies F = BIL\sin\theta$$

In the above, we must stress that the magnetic force is acting on the **electrons** flowing through the conductor. Particularly when the magnetic force is perpendicular to the direction of electron flow, the force on these electrons is *transmitted to the wire* upon collision.

The magnetic force acting on a **moving charged particle** of charge Q and speed v is given by

$$\vec{\mathbf{F}} = \vec{\mathbf{B}} \times q\vec{\mathbf{v}} \implies F = Bqv\sin\theta$$

16.3.1 Parallel Current-Carrying Conductors

If two long wires run parallel to, and near each other,

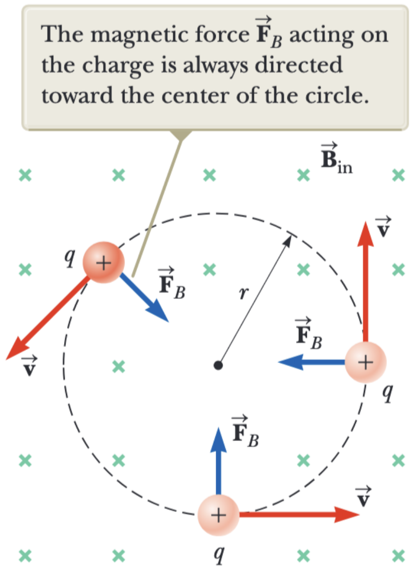
- if both their currents flow in the **same direction**, the wires will **attract** each other;
- if their currents flow in **opposite directions**, the wires will **repel** each other.

16.3.2 Motion of Charged Particles

When a charged particle is projected at a right angle into a magnetic field, notice that the magnetic force F must always be perpendicular to the particle's velocity v.

Since, the distance travelled in the direction of the force is zero, *magnetic force does no work on a charged particle.*

To put it more specifically, since F is of constant magnitude and is always perpendicular to v, the charged particle undergoes *circular motion.*



If the magnetic field is pointing into the paper,

- positively charged particles travel counterclockwise,
- negatively charged particles travel clockwise.

The opposite holds true if the magnetic field is pointing out of the paper instead.

Now, we will analyse the Newtonian mechanics behind the circular motion caused by magnetic forces. By Newton's second law,

$$\sum \vec{\mathbf{F}} = \vec{\mathbf{F}}_B = m\vec{\mathbf{a}}$$

Since the particle undergoes circular motion, we can treat the magnetic force as a centripetal force providing a corresponding centripetal acceleration.

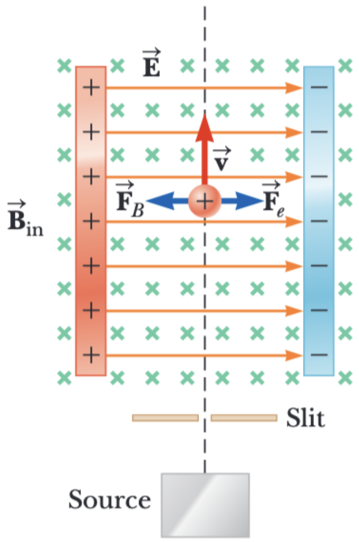
$$\vec{\mathbf{F}}_B = Bq\vec{\mathbf{v}} = \frac{m\vec{\mathbf{v}}^2}{r}$$

From this equation, we can obtain various quantities.

$$r = \frac{mv}{qB} \implies \omega = \frac{qB}{m} \implies T = \frac{2\pi m}{qB}$$

16.3.3 Velocity Selector

A **crossed-field velocity selector** consists of a magnetic field and an electric field applied over the same region. This results in all undeflected particles passing through the region *to have the same velocity.*



As a charged particle enters the region, it will experience an electric force $\vec{\mathbf{F}}_E$ and a magnetic force $\vec{\mathbf{F}}_B$ simultaneously.

In order for the particle to not be deflected, it must be in equilibrium. In other words, $\vec{\mathbf{F}}_E = \vec{\mathbf{F}}_B$.

$$\vec{\mathbf{F}}_E = \vec{\mathbf{F}}_B \implies Bqv = qE$$

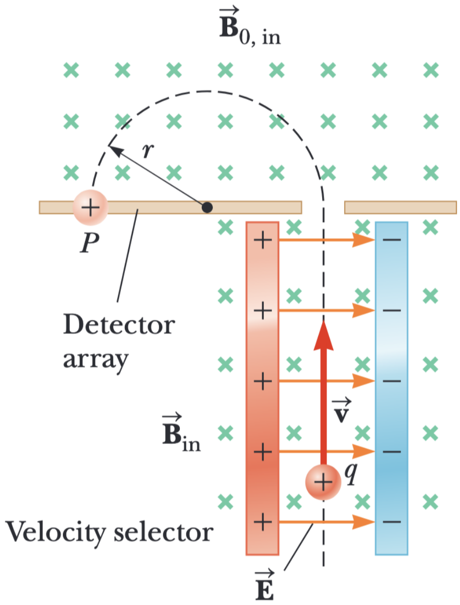
$$\implies v = \frac{E}{B}$$

Only particles having the speed $v = E/B$ will pass undeflected through the crossed-field region.

- If particles move with higher velocity v, the magnetic force becomes stronger than the electric force, causing leftwards deflection.
- If particles move with lower velocity v, the magnetic force becomes weaker than the electric force, causing rightwards deflection.

16.3.4 Mass Spectrometer

Some questions may include a **mass spectrometer**, where charged particles are first sent through a velocity selector, before being shot out into a region of uniform magnetic field that causes it to move in a *semicircular path.*



The velocity of the particles after passing through the selector is $v = E/B$.

Upon entering the new magnetic field $\vec{\mathbf{B}}_0$, the charges undergo circular motion, moving in a semicircular of radius r before hitting a flat plane.

$$r = \frac{mv}{qB} \implies \frac{m}{q} = \frac{rB_0}{v} = rB_0 \left(\frac{B}{E}\right)$$

The above result provides us with a relation between the specific charge m/q, the radius r, as well as the magnitudes of each field strength.

17 Electromagnetic Induction

17.1 Magnetic Flux

Definition 17.1: Magnetic Flux

Magnetic flux is defined as the product of an area and the component of the magnetic flux density perpendicular to that area.

$$\Phi = \mathbf{B} \cdot \mathbf{A} = BA\cos\theta$$

where Φ is the magnetic flux for a uniform magnetic flux density B at an angle θ to the normal of an area A.

The S.I. unit of magnetic flux is the *weber* (wb).

Definition 17.2: Magnetic Flux Linkage

Magnetic flux linkage is the product of the magnetic flux Φ passing through the coil and the number of turns N on the coil.

$$N\Phi = NBA\cos\theta$$

The magnetic flux linkage is simply the total magnetic flux passing through an area A bounded by a coil with N turns.

17.2 Faraday's Law

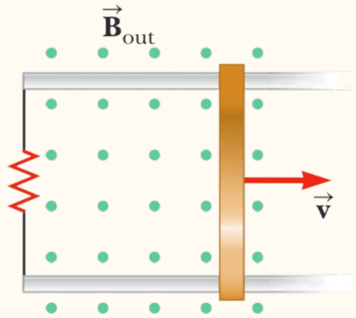
Definition 17.3: Faraday's Law

Faraday's law of electromagnetic induction states that the magnitude of the induced e.m.f. is **directly proportional** to the **rate of change** of the magnetic flux linkage.

$$|\epsilon| = \frac{d(N\Phi)}{dt} = \frac{d(NBA\cos\theta)}{dt}$$

Example 17.1

A rod of length l moves at a constant speed of v perpendicular to a magnetic field of flux density B. Use Faraday's law to determine the e.m.f. induced across the ends of the wire.



$$|\epsilon| = \frac{\Delta(NBA\cos\theta)}{\Delta t} = B\frac{\Delta A}{\Delta t}$$
$$= Blv$$

Example 17.2

A disc of area A perpendicular to the uniform magnetic field is rotated with a constant frequency f. Determine the e.m.f. induced between the centre and the rim of the disc.

We can regard the disc as a many-spoked wheel, where the radius of the disc sweeps through the magnetic field.

$$|\epsilon| = \frac{\Delta(NBA\cos\theta)}{\Delta t} = B\frac{\pi r^2}{T}$$
$$= B\pi r^2 f$$

17.3 Lenz's Law

Definition 17.4: Lenz's Law

Lenz's law states that the induced current is in a direction so as to produce effects which oppose the change in the magnetic flux.

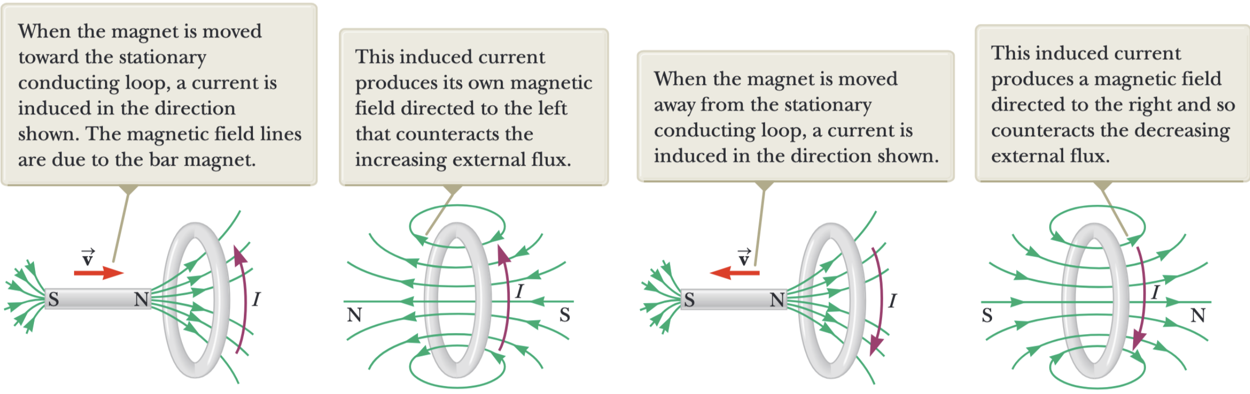
$$\epsilon = -\frac{d(NBA\cos\theta)}{dt}$$

In a sense, the induced current flows in such a way that it attempts to preserve the original magnetic flux. If the magnetic flux is decreasing, the induced current flows to increase it. If the magnetic flux is increasing, the induced current flows to decrease it.

Considering the slidewire generator shown in Example 17.1, as the rod moves outwards, the magnetic flux directed into the area enclosed by the circuit *increases with time.*

Due to Lenz's law, the induced current flows to oppose this change. It must produce a field *directed out of the page.* Hence, the induced current must be directed *counterclockwise* when the rod moves to the right.

Similarly, the following figure shows the flow of induced current when a bar magnet moves towards and away from a conducting loop.

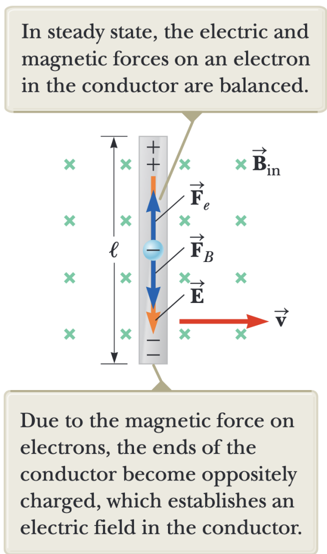


17.4 Motional Electromotive Force

Motional e.m.f. refers to the electromotive force induced in a conductor that moves through a constant magnetic field.

Consider a straight current-carrying conductor of length ℓ moving uniformly with velocity $\vec{\mathbf{v}}$ in a direction perpendicular to a magnetic field $\vec{\mathbf{B}}$.

The electrons in the conductor experience a force $\vec{\mathbf{F}}_b = \vec{\mathbf{B}} \times q\vec{\mathbf{v}}$ directed along the length of the conductor — electrons thus accumulate at one end of the conductor, **creating an electric field** $\vec{\mathbf{E}}$ inside the conductor.



In order for the velocity to remain constant, the forces must balance.

$$qE = Bvq \implies E = vB$$

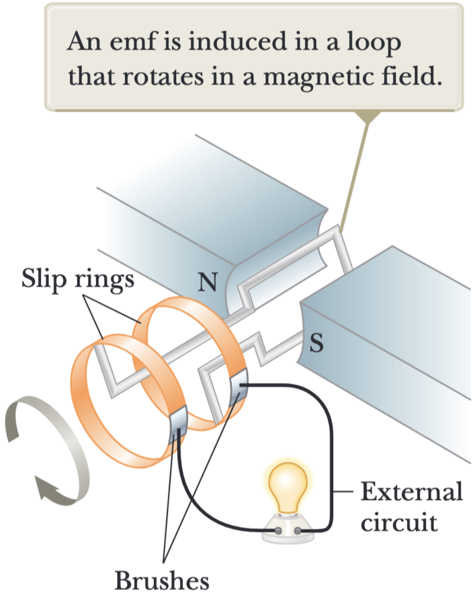
The magnitude of the electric field produced in the conductor is related to the potential difference across its ends, where $\Delta V = E\ell$. This gives

$$\Delta V = E\ell = B\ell v$$

Hence, a **potential difference is maintained** as long as the conductor continues to move across the field and “cut through” the magnetic flux.

17.5 Generators

The **alternating-current generator** is a loop of wire that rotates due to the magnetic force exerted on it by a magnetic field.



As the loop rotates, the magnetic flux Φ changes with time, which induces an electromotive force and a current. Assuming that the generator rotates with constant angular speed ω,

$$\Phi = BA\cos\theta = BA\cos\omega t$$

The induced electromotive force in the coil is thus

$$\begin{aligned} \epsilon &= -\frac{d(N\Phi)}{dt} = -NBA\frac{d}{dt}(\cos\omega t) \\ &= NBA\,\omega\sin\omega t \end{aligned}$$

The electromotive force **varies sinusoidally** with time. The maximum electromotive force has the value $NBA\,\omega$, when ωt is either 0^{o} or 180^{o} — that is, when $\vec{\mathbf{B}}$ is perpendicular to the plane of the coil.

Example 17.3

The coil in an AC generator consists of 8 turns of wire, each of area $A = 0.09\ \text{m}^2$. The coil rotates in a magnetic field of flux density 0.5 T at a constant frequency of 60.0 Hz.

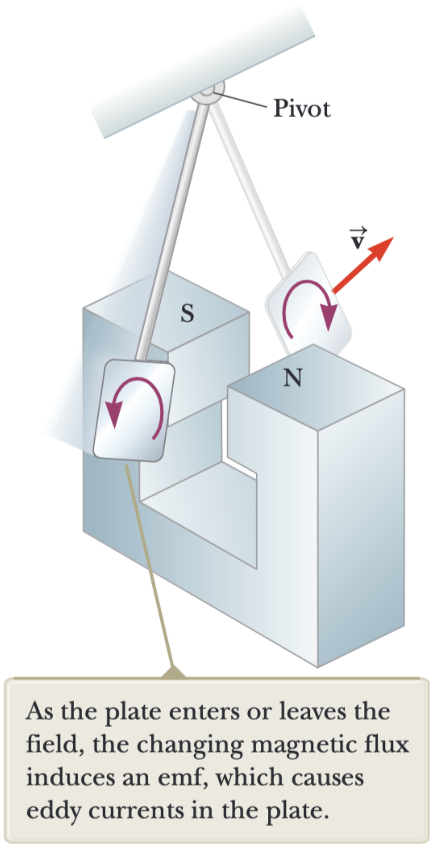
Find the maximum induced electromotive force in the coil.

The induced electromotive force is at a maximum when the magnetic field is in the plane of the coil.

$$\epsilon = \frac{d(N\phi)}{dt} \implies \epsilon_{\text{max}} = NBA\,\omega$$
$$\implies \epsilon_{\text{max}} = 8(0.50)(0.09) \times (2\pi)(60) = \boxed{136 \text{ V}}$$

17.6 Eddy Currents

If a solid plate conductor is subjected to a changing magnetic flux, **eddy currents** will flow simultaneously along many different paths in swirls along the conductor's plane.



In the diagram shown above, as the plate enters the magnetic field, the plate cuts the magnetic flux, which induces an e.m.f. in the plate. However, **the rate of cutting varies over the whole plate**, so different electromotive forces are induced in different parts of the plate, leading to the formation of eddy currents.

The direction of the eddy currents can be derived using Lenz"s law. For example, as the plate enters the field in the direction of $\vec{v}$ shown above, the field is going into the face of the plate, hence the eddy current must provide its own magnetic field out of the page. This causes the eddy current to flow counterclockwise.

Heat energy is dissipated as eddy currents flow in the conductor.

Eddy currents can be **reduced** by eliminating paths for current flow, through

- cutting slots in the plate, or

- laminating the conductor (i.e. built up in thin layers separated by a non-conducting material such as lacquer) to prevent large current loops and confine the currents to small loops in individual layers.

Part VI

H3 Physics

A1 Inertial Frames

A1.1 Frames of Reference

Frames of reference are *perspectives of motion*. There exists a stationary reference frame in which stationary observers undergo no movement relative to Earth's rotation. For observers that moving at a constant speed, they are within an *inertial reference frame*, where the frame's velocity may be different from the stationary frame, but exhibits zero acceleration.

Definition A1.1: Frame of Reference

A **frame of reference** can be uniquely defined by the following four-vector.

$$(x, y, z, t)$$

where x, y, z uniquely define the 3-dimensional coordinates of a body, and t refers to the time coordinate.

Definition A1.2: Galilean Relativity

The **principle of Galilean relativity** (and also **Einstein's first postulate**) states that the laws of mechanics must be the same in all inertial frames of reference.

For inertial frames of reference, a body that is not acted on by an external force will continue in its state of constant motion or rest.

A1.1.1 Galilean Transformations

The Galilean transformation relates coordinates measured in one inertial reference frame $S \mapsto (x, y, z, t)$ to that measured in other inertial frame $S' \mapsto (x', y', z', t')$.

For example, if the S' frame is moving at a constant velocity u solely applied to the x-direction, the S' observer will perceive the stationary body in frame S to be moving in the opposite direction. Therefore, we obtain

$$(x', y', z', t') = (x - ut, y, z, t)$$

We are able to employ a generaliasation of the Galilean transformation equations, which provides us with the following two equations.

$$\begin{aligned} \vec{\mathbf{r}}' &= \vec{\mathbf{r}} - \vec{\mathbf{v}}t \\ t' &= t \end{aligned}$$

where $\vec{\mathbf{v}}$ is the relative velocity of the two inertial reference frames S and S'. Across all frames of reference, *time is invariant*, therefore $t' = t$.

For an object moving in the stationary frame S, its velocity as measured by another observer in the same frame S will be $v_r = dr/dt$. We also note that its velocity v_r' as measured by an observer in the comoving S' frame. Differentiating the Galilean transformation equations gives us

$$\frac{dr'}{dt} = \frac{dr}{dt} - u_r \implies v_r' = v_r - u_r$$

A1.2 Center of Mass

There is a special point in a mechanical system, termed the **center of mass**, that moves as if all of the mass of the system were concentrated at that one point.

Consequently, the system moves as if an external force were applied to a single particle of mass M (where M is the total mass of the system) located the center of mass.

For a system of discrete particles, the center of mass is defined to be

$$x_{\text{CM}} = \frac{1}{M}\sum_i m_i x_i \qquad y_{\text{CM}} = \frac{1}{M}\sum_i m_i y_i \qquad z_{\text{CM}} = \frac{1}{M}\sum_i m_i z_i$$

For an extended object with a total mass M, we can model it as a system containing a large number of infinitesimally small mass elements. For the x-axis, this gives us

$$x_{\text{CM}} = \lim_{\Delta m_i \to 0} \frac{1}{M}\sum_i x_i \Delta m_i \implies x_{\text{CM}} = \frac{1}{M}\int x\,dm$$

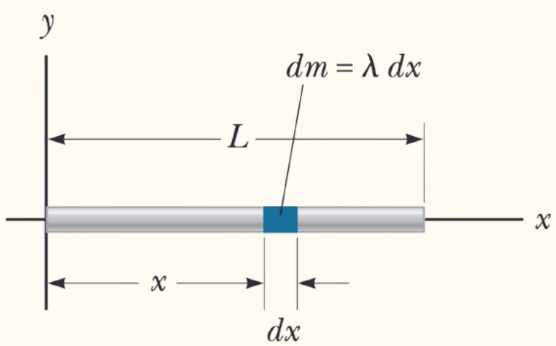
For all the three-dimensional axes, we obtain

$$x_{\text{CM}} = \frac{1}{M}\int x\,dm \qquad y_{\text{CM}} = \frac{1}{M}\int y\,dm \qquad z_{\text{CM}} = \frac{1}{M}\int z\,dm$$

Note that the center of mass *need not be within the object itself.*

Example A1.1

Find the center of mass of a uniform rod with mass M and length L.



Let the linear mass density be $\lambda = M/L$. If the rod is divided into elements dx,

$$dm = \lambda\,dx$$

Finding the center of mass,

$$\begin{aligned} x_{\text{CM}} &= \frac{1}{M}\int x\,dm = \frac{1}{M}\int_0^L x\lambda\,dx \\ &= \frac{\lambda}{M}\left[\frac{x^2}{2}\right]_0^L = \frac{\lambda L^2}{2M} = \frac{M}{L}\times\frac{L^2}{2M} \\ &= \boxed{\frac{1}{2}L} \end{aligned}$$

Example A1.2

Suppose a rod of length L is non-uniform such that its mass per unit length varies linearly with x according to the expression $\lambda = kx$, where k is a constant.

Show that the center of mass of the rod is $\frac{2}{3}L$ away from the lighter end.

Expressing the infinitesimal mass element for a length element dx,

$$dm = \lambda\,dx = kx\,dx$$

Since the rod has non-uniform mass, we integrate dm throughout to obtain M.

$$M = \int dm = \int_0^L kx\,dx = \frac{kL^2}{2}$$

Finding the center of mass,

$$\begin{aligned} x_{\text{CM}} &= \frac{1}{M}\int x\,dm = \frac{k}{M}\int_0^L x^2\,dx \\ &= \frac{k}{M}\times\left[\frac{x^3}{3}\right]_0^L = \frac{2k}{kL^2}\times\frac{L^3}{3} \\ &= \boxed{\frac{2}{3}L} \end{aligned}$$

Example A1.3

Find the center of mass of a flat, semi-circular object of radius R and uniform density.

Since the semicircle is symmetrical about the y-axis, $x_{\text{CM}} = 0$.

We divide the semicircle into horizontal strips of length dy and width $2\sqrt{R^2 - y^2}$ each. Let the area mass density be denoted by $\sigma = M/A$.

$$\begin{aligned} dm &= \sigma\,dA = \sigma \times x \times dy \\ &= \sigma \times 2\sqrt{R^2 - y^2} \times dy \end{aligned}$$

Finding y_{CM},

$$\begin{aligned} y_{\text{CM}} = \frac{1}{M}\int_0^R y\,dm &= \frac{1}{M}\int_0^R y\,\sigma\,2\sqrt{R^2-y^2}\,dy \\ &= -\frac{2\sigma}{3M}\int_0^R -3y\sqrt{R^2-y^2}\,dy \\ &= -\frac{2\sigma}{3M}\left[(R^2-y^2)^{(3/2)}\right]_0^R \\ &= \frac{2\sigma}{3M}(R^3) \end{aligned}$$

Substituting $\sigma = M/A = M/(\frac{1}{2}\pi R^2)$,

$$y_{\text{CM}} = \frac{4}{3\pi R^2}(R^3) = \boxed{\frac{4R}{3\pi}}$$

A1.2.1 Velocity & Momentum

Where the total mass M of the system remains constant, we can describe the motion of the system in terms of the velocity and acceleration of the center of mass of the system.

$$\vec{\mathbf{v}}_{\text{CM}} = \frac{d\vec{\mathbf{r}}_{\text{CM}}}{dt} = \frac{1}{M}\sum_i m_i\vec{\mathbf{v}}_i$$

The total momentum of the system is equal to the total mass multiplied by the velocity of the center of mass.

$$\vec{\mathbf{p}}_{\text{tot}} = \sum_i \vec{\mathbf{p}}_i = \sum_i m_i\vec{\mathbf{v}}_i = M\vec{\mathbf{v}}_{\text{CM}}$$

This states that the total momentum *is equal to* the total mass times the velocity of the center of the mass.

A1.2.2 Force & Acceleration

The acceleration of the center of mass is given by

$$\vec{\mathbf{a}}_{\text{CM}} = \frac{d\vec{\mathbf{v}}_{\text{CM}}}{dt} = \frac{1}{M}\sum_i m_i\vec{\mathbf{a}}_i \implies M\vec{\mathbf{a}}_{\text{CM}} = \sum_i \vec{\mathbf{F}}_i$$

Summing over all internal forces, they cancel in pairs, and the net force on the system is caused only by external forces.

$$\sum \vec{\mathbf{F}}_{\text{ext}} = M\vec{\mathbf{a}}_{\text{CM}} = \frac{d\vec{\mathbf{p}}}{dt}$$

When a system of particles is acted on by external forces, the center of mass moves just as though all the mass were concentrated at that point and it were acted on by a net force equal to the sum of external forces on the system.

If the net external force acting on the system is zero, then we see that

$$\frac{d\vec{\mathbf{p}}}{dt} = 0 \implies M\vec{\mathbf{v}}_{\text{CM}} = \vec{\mathbf{p}} = \text{constant}$$

A1.2.3 Center of Mass Frame

The **center of mass frame**, also known as the **zero momentum frame**, is a useful tool in solving collision problems. This frame is chosen such that the total momentum is zero, when the inertial frame *travels at the same velocity as the center of mass.*

Take two objects of masses m_1, m_2 and v_1, v_2 respectively in the stationary frame.

Then, for the center of mass frame,

$$v_1' = v_1 - v_{\mathrm{CM}} \qquad v_2' = v_2 - v_{\mathrm{CM}}$$

Since $\sum \vec{\mathbf{p}} = 0$ for the center of mass frame, this quality implies that

$$p_{1,i} = -p_{2,i} \qquad p_{1,f} = -p_{2,f}$$

For elastic collisions, kinetic energy is also conserved. Therefore,

$$\frac{{p_{1,i}}^2}{2m_1} + \frac{{p_{2,i}}^2}{2m_2} = \frac{{p_{1,f}}^2}{2m_1} + \frac{{p_{2,f}}^2}{2m_2}$$

Combining these two conditions, we find that only 1 solution exists, that is when the *final velocity is opposite of the initial velocity.*

$$v_f' = -v_i'$$

A2 Rotational Motion

A summary of the analogues between rotational and translational motion is shown below.

Rotational Motion About a Fixed Axis	**Translational Motion**
Angular speed $\omega = d\theta/dt$	Translational speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Translational acceleration $a = dv/dt$
Net torque $\Sigma\tau_{\text{ext}} = I\alpha$	Net force $\Sigma F = ma$
If $\alpha =$ constant $\begin{cases} \omega_f = \omega_i + \alpha t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If $a =$ constant $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$	Work $W = \int_{x_i}^{x_f} F_x \, dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $P = \tau\omega$	Power $P = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Net torque $\Sigma\tau = dL/dt$	Net force $\Sigma F = dp/dt$

A2.1 Angular Quantities

For rotational dynamics, we focus on **rigid objects** — non-deformable objects, where relative locations of the object's particles remain constant.

Parallel to translational kinematics, we define angular **displacement**, **velocity**, and **acceleration**

For an object rotating through a circular path with radius r, the arc length s and the angle θ it moves through is related by

$$s = r\theta \implies \theta = \frac{s}{r}$$

Angular displacement of the rigid object is thus

$$\Delta\theta = \theta_f - \theta_i$$

The **average angular speed** $\langle\omega\rangle$ is simply angular displacement over a time interval Δt.

$$\langle\omega\rangle = \frac{\Delta\theta}{\Delta t}$$

The **instantaneous angular speed** ω is defined as the limit of the average angular speed as $\Delta t \to 0$.

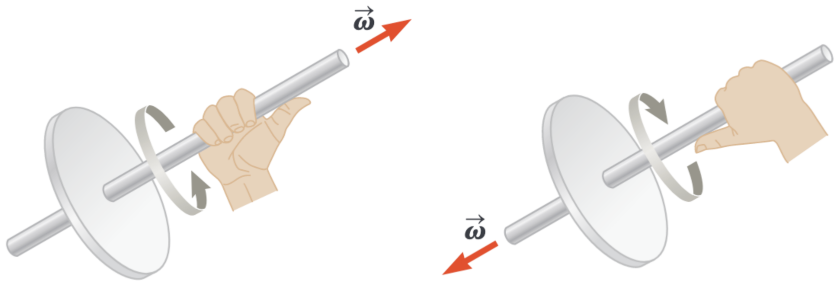
$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

The same ideas hold for **angular acceleration**.

$$\langle\alpha\rangle = \frac{\Delta\omega}{\Delta t} \implies \alpha = \frac{d\omega}{dt}$$

Thus far, ω and a merely represent the magnitudes of the angular velocity and angular acceleration vectors $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\alpha}}$.

For rotation about a fixed axis, the only direction that uniquely specifies the rotational motion is the **direction along the axis of rotation**. The direction of both $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\alpha}}$ (since the direction of $\vec{\boldsymbol{\alpha}}$ directly follows from that of $\vec{\boldsymbol{\omega}}$) can be found using the right-hand rule.



A2.1.1 Relation to Translational Quantities

For a rotating rigid object, translational velocity is always tangential to the circular path (and is hence called **tangential velocity**).

Since velocity is given by the rate of change of displacement,

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} \implies v = r\omega$$

The tangential speed of a point on a rotating rigid object is equal to the perpendicular distance of that point from the axis of rotation multiplied by the angular speed.

Hence, though every point on the rigid object has the **same angular speed**, they may **differ in tangential speed**, simply because r is not the same for all points on the object.

Tangential speed *increases* as one moves *outward* from the centre of rotation — the outer end of a rotating CD moves faster than a point near the centre.

We can relate the angular acceleration of the rotating rigid object to the **tangential acceleration** of any point by taking the time derivative of v.

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} \implies a_t = r\alpha$$

The expression for centripetal acceleration is standard.

$$a_c = \frac{v^2}{r} = v\omega = r\omega^2$$

The **total acceleration vector** at any point is thus

$$\begin{aligned} \vec{\boldsymbol{a}} = \vec{\boldsymbol{a}}_t + \vec{\boldsymbol{a}}_c &\implies a = \sqrt{{a_t}^2 + {a_c}^2} \\ &\implies a = \sqrt{r^2\alpha^2 + r^2\omega^4} \\ &\implies a = r\sqrt{\alpha^2 + \omega^4} \end{aligned}$$

A2.1.2 Rotational Kinematics

We consider the special case of **a rigid object under constant angular acceleration**.

Recalling the equation for instantaneous angular acceleration and integrating from $t_i = 0$ to $t_f = t$, we can obtain an equation that determines the *angular speed* ω_f of the object *at any later time* t.

$$\begin{aligned}
\alpha = \frac{d\omega}{dt} &\implies d\omega = \alpha \, dt \\
&\implies \int_0^t 1 \, d\omega = \int_0^t \alpha \, dt \\
&\implies \boxed{w_f = w_i + \alpha t} \quad \text{(for constant } \alpha\text{)}
\end{aligned}$$

Substituting this expression into our previous derivation for instantaneous ω, we obtain an expression relating the *angular displacement at any time* t in relation to initial displacement, velocity, and acceleration.

$$\begin{aligned}
\omega = \frac{d\theta}{dt} &\implies \omega_i + \alpha t = \frac{d\theta}{dt} \\
&\implies \int_0^t (\omega_i + \alpha t) \, dt = \int_0^t \frac{d\theta}{dt} \, dt \\
&\implies \omega_i t + \frac{1}{2}\alpha t^2 = \theta_f - \theta_i \\
&\implies \boxed{\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2} \quad \text{(for constant } \alpha\text{)}
\end{aligned}$$

Making t the subject in the first equation and substituting it into the second equation to eliminate t, we obtain

$$\begin{aligned}
\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 &\implies \theta_f = \theta_i + \omega_i \left(\frac{\omega_f - \omega_i}{\alpha} \right) + \frac{1}{2}\alpha \left(\frac{\omega_f - \omega_i}{\alpha} \right)^2 \\
&\implies 2\alpha(\theta_f - \theta_i) + 2\omega_i(\omega_f - \omega_i) + (\omega_f - \omega_i)^2 \\
&\implies 2\alpha(\theta_f - \theta_i) = {\omega_f}^2 - {\omega_i}^2 \\
&\implies \boxed{{\omega_f}^2 = {\omega_i}^2 + 2\alpha(\theta_f - \theta_i)} \quad \text{(for constant } \alpha\text{)}
\end{aligned}$$

Eliminating α instead,

$$\begin{aligned}
\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 &\implies \theta_f = \theta_i + \omega_i t + \frac{1}{2} \left(\frac{\omega_f - \omega_i}{t} \right) t^2 \\
&\implies \boxed{\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t} \quad \text{(for constant } \alpha\text{)}
\end{aligned}$$

This set of equations is *analogous* to the *kinematic equations of motion* for translational motion.

A2.2 Moment of Inertia

Definition A2.1: Moment of Inertia

The **moment of inertia** is a quantity of an object that is directly proportional to the masses of its particles and their distances from the axis of rotation squared.

For a *single particle*, its moment of inertia I is given as

$$I = mr^2$$

where m is the mass of the particle, and r is the distance from the particle to its axis of rotation.

An body's moment of inertia is **analogous to mass** for translational motion, for it is the *resistance* to chnages in rotational motion.

The calculation of the moment of inertia for extended objects becomes a bit more complicated. The figure below shows the moments of inertia for certain homogenous rigid objects with varying geometries.

A2.2.1 Calculation

The moment of inertia of a system of discrete particles is simply the sum of the moments of inertia of all its particles.

For a continuous rigid object, we imagine the object being divided into many small elements of mass Δm_i, then take the limit.

$$I = \lim_{\Delta m_i \to 0} \sum_i {r_i}^2 \Delta m_i = \boxed{\int r^2 \, dm}$$

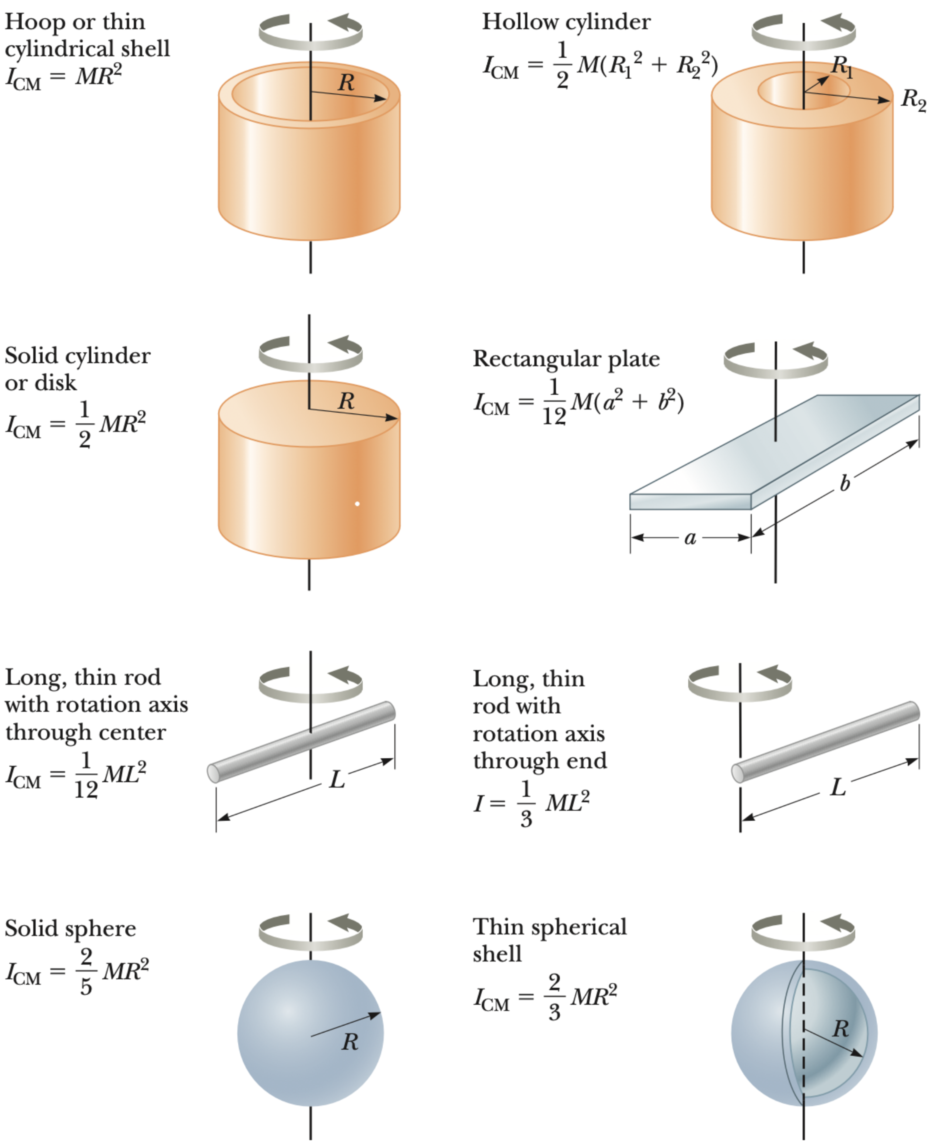
It is usually easier to calculate moments of inertia in terms of volume. Expressing density in volumetric mass form as $\rho = m/V$, we see that $dm = \rho \, dV$, hence

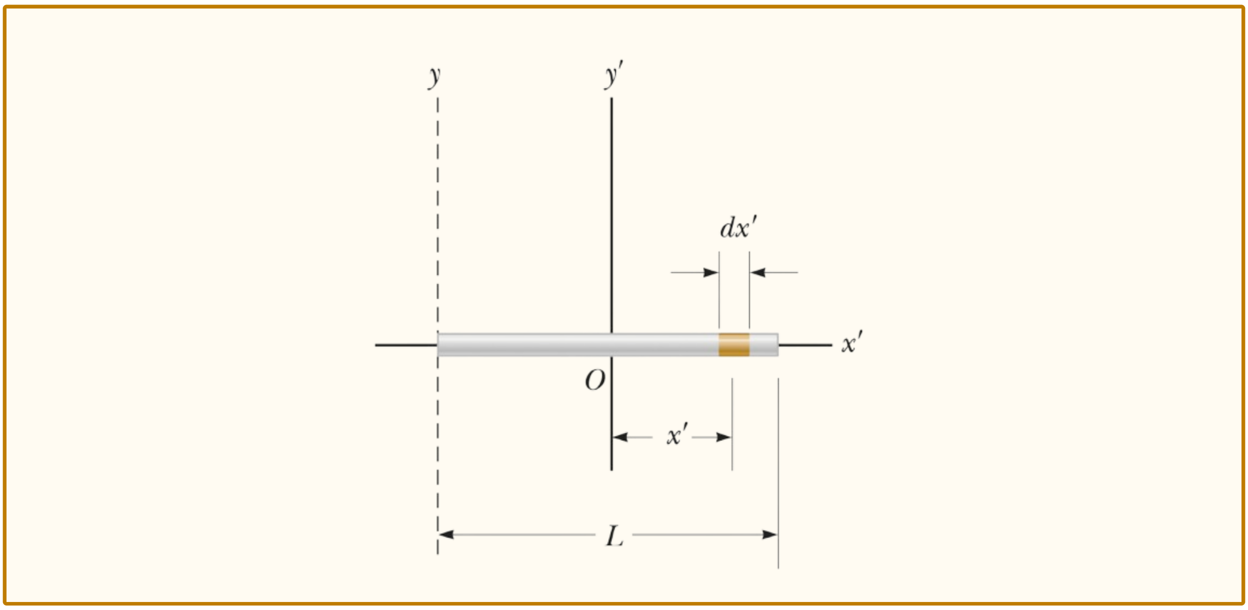
$$I = \int \rho r^2 \, dV$$

If the object is *homogenous*, ρ is *constant*, and the integral can be evaluated for a known geometry.

Example A2.1

Calculate the moment of inertia of a uniform thin rod of length L and mass M about an axis perpendicular to the rod and passing through its center of mass.


Hoop or thin cylindrical shell
$I_{CM} = MR^2$
R
Hollow cylinder
$I_{CM} = \frac{1}{2}M(R_1{}^2 + R_2{}^2)$
R_1
R_2
Solid cylinder or disk
$I_{CM} = \frac{1}{2}MR^2$
R
Rectangular plate
$I_{CM} = \frac{1}{12}M(a^2 + b^2)$
b
a
Long, thin rod with rotation axis through center
$I_{CM} = \frac{1}{12}ML^2$
L
Long, thin rod with rotation axis through end
$I = \frac{1}{3}ML^2$
L
Solid sphere
$I_{CM} = \frac{2}{5}MR^2$
R
Thin spherical shell
$I_{CM} = \frac{2}{3}MR^2$
R


y
y′
dx′
x′
O
x′
L

The shaded length dx has a mass dm equal to the mass per unit length λ multiplied by dx.

$$dm = \lambda\, dx = \frac{M}{L}\, dx$$

Substituting,

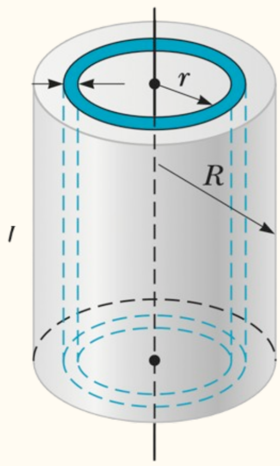
$$I = \int r^2\, dm = \int_{-L/2}^{L/2} (x)^2 \frac{M}{L}\, dx = \frac{M}{L} \int_{-L/2}^{L/2} (x)^2\, dx$$

$$= \frac{M}{L} \left[\frac{(x)^3}{3} \right]_{-L/2}^{L/2}$$

$$= \boxed{\frac{1}{12} M L^2}$$

Example A2.2

A uniform solid cylinder has a radius R, mass M, and length L. Calculate its moment of inertia about its central axis.



Here, we divide the cylinder into many cylindrical shells, each having radius r, thickness dr, and height L.

The density of the cylinder is ρ and the volume dV of each shell is given by

$$dV = L\, dA = L(2\pi r)\, dr$$

Expressing dm in terms of dr,

$$dm = \rho\, dV = \rho L(2\pi r)\, dr$$

Substituting,

$$I = \int r^2\, dm = \int r^2 \left(pL(2\pi r)\, dr\right) = 2\pi\rho L \int_0^R r^3\, dr = \frac{1}{2}\pi\rho L R^4$$

Using the total volume to express density,

$$\rho = \frac{M}{V} = \frac{M}{\pi R^2 L}$$

Finishing off,

$$I = \frac{1}{2}\pi\left(\frac{M}{\pi R^2 L}\right) LR^4 = \boxed{\frac{1}{2}MR^2}$$

A2.2.2 Parallel-Axis Theorem

Definition A2.2: Parallel-Axis Theorem

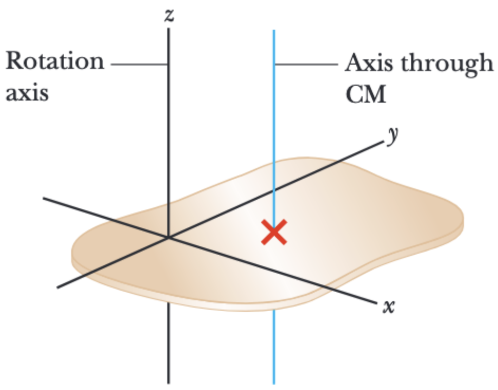
The **parallel-axis theorem** states that for a body of mass m with a moment of inertia I_{CM} when rotating about its center of mass, its moment of inertia I for when rotating about a parallel axis a distance d away from the center of mass axis is

$$I = I_{\text{CM}} + md^2$$

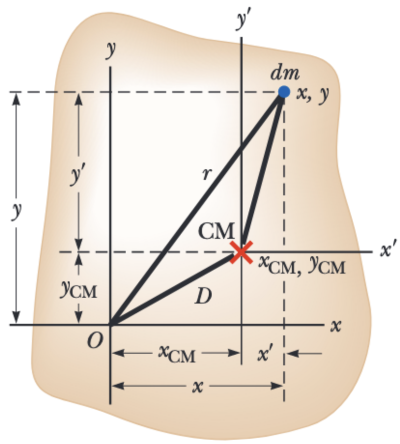
This is used to simplify calculations of moments of inertia for objects rotating around an arbitrary axis.

Proof. We note that the moment of inertia does not depend on the distribution of mass along the axis — as seen in the case of the cylinder, its moment of inertia is *independent of its length*.

As such, as long as the object is homogenous and symmetric, we can simply consider a cross-section of the object lying on a 2-dimensional plane, as shown below.



Now, we consider a mass element dm.



As this element is a distance $r = \sqrt{x^2 + y^2}$ from the axis of rotation (which passes through the origin O), the moment of inertia of the entire object is given by

$$I = \int r^2 \, dm = \int (x^2 + y^2) \, dm$$

As a trick, we can relate the coordinates of dm to the object's centre of mass, where $x = x' + x_{\text{CM}}$ and $y = y' + y_{\text{CM}}$ respectively as shown by the figure above. Therefore,

$$\begin{aligned} I &= \int \left[(x' + x_{\text{CM}})^2 + (y' + y_{\text{CM}})^2\right] \, dm \\ &= \int \left[(x')^2 + (y')^2\right] \, dm + 2x_{\text{CM}} \int x' \, dm + 2y_{\text{CM}} \int y' \, dm + ({x_{\text{CM}}}^2 + {y_{\text{CM}}}^2) \int dm \\ \therefore \; I &= \boxed{I_{\text{CM}} + md^2} \end{aligned}$$

where m is the mass of the object and d is the perpendicular distance between the given axis and the axis through the centre of mass. □

A2.3 Torque

Definition A2.3: Torque

Torque, denoted by $\boldsymbol{\tau}$, is the cross product between a force $\vec{\mathbf{F}}$ applied to a body and the body's perpendicular distance $\vec{\mathbf{r}}$ to its axis of rotation.

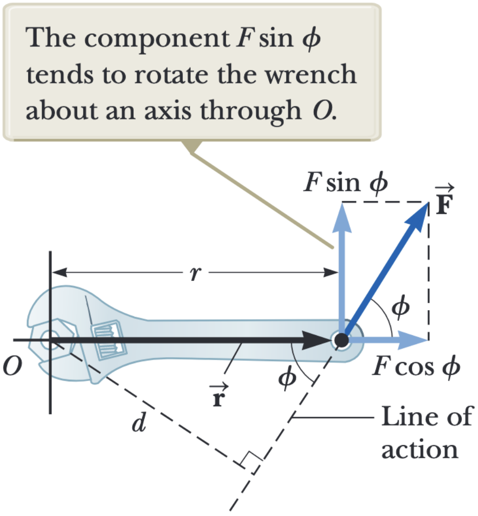
$$\boldsymbol{\tau} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}$$

Torque acts as the rotational analogue to linear force.

When a force is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis. The tendency of a force to rotate an object about some axis is measured by torque $\vec{\boldsymbol{\tau}}$.

Considering the wrench above, the force $\vec{\boldsymbol{F}}$ acts at an angle ϕ to the length of the wrench $\vec{\boldsymbol{r}}$. We define the **magnitude of the torque** associated with $\vec{\boldsymbol{F}}$ around the axis O by the expression

$$\tau = |\vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}| = rF \sin\phi$$



A2.3.1 Rigid Object Under Net Torque

Consider a particle of mass m rotating in a circle of radius r under the influence of a tangential net force $\sum \vec{\mathbf{F}}_t$ and a radial net force for $\sum \vec{\mathbf{F}}_r$. This tangential net force causes a tangential acceleration $\vec{\mathbf{a}}_t$, whereas the radial net force provides the centripetal acceleration for the circular motion.

By Newton's second law, examining the tangential acceleration,

$$\sum F_t = ma_t$$

The magnitude of the net torque due to $\sum \vec{\mathbf{F}}_t$ on the particle about an axis perpendicular to the page through the centre of the circle is

$$\sum \tau = \sum F_t \, r = (ma_t)\, r$$

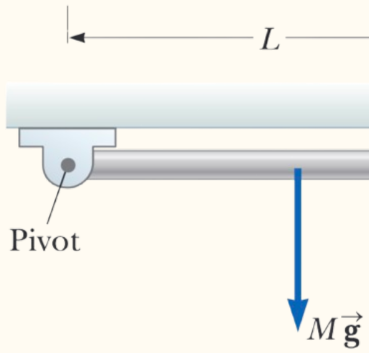
As mentioned earlier in the *relation between angular and translational quantities*, $a = r\alpha$, hence

$$\sum \tau = (ma_t)\, r = (mr\alpha)\, r = (mr^2)\, \alpha \implies \boxed{\sum \tau = I\alpha}$$

The above statement is analogous to $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$.

Example A2.3

A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane.



The rod is released from rest in the horizontal position. What is the initial angular acceleration of the rod and the initial translational acceleration of its right end?

By taking moments about the pivot,

$$\sum \tau = F \cdot r = Mg\frac{L}{2}$$

Since $\sum \tau = I\alpha$,

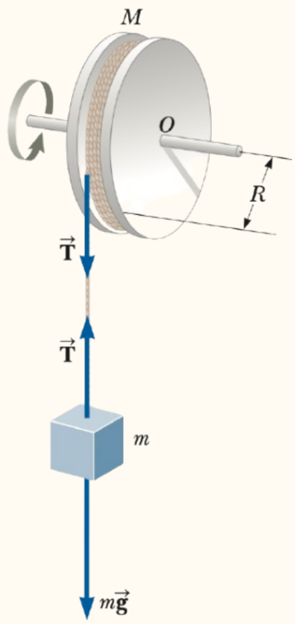
$$\alpha = \frac{\sum \tau}{I} = \frac{Mg(L/2)}{\frac{1}{3}ML^2} = \boxed{\frac{3g}{2L}}$$

$$a_t = L\alpha = \boxed{\frac{3}{2}g}$$

Example A2.4

A wheel of radius R, mass M and moment of inertia I is mounted on a frictionless, horizontal axle.

A light cord wrapped around the wheel supports an object of mass m. When the wheel is released, the object accelerates downward, the cord unwraps off the wheel, and the wheel rotates with an angular acceleration.



Find expressions for

1. the angular acceleration of the wheel,
2. the translational acceleration of the object, and
3. the tension in the cord.

From $\sum \tau = I\alpha$,

$$\alpha = \frac{\sum \tau}{I} = \frac{TR}{I}$$

Applying Newton's second law,

$$mg - T = ma \implies a = \frac{mg - T}{m}$$

Due to conservation of string, the translational acceleration of the object is equal to the tangential acceleration of the cord.

$$a = R\alpha \implies \frac{mg - T}{m} = \frac{TR^2}{I}$$

$$\implies T = \boxed{\frac{mg}{1 + (mR^2/I)}}$$

$$\implies a = \boxed{\frac{g}{1 + (I/mR^2)}}$$

$$\implies \alpha = \frac{a}{R} = \boxed{\frac{g}{R + (I/mR)}}$$

A2.4 Angular Momentum

Definition A2.4: Angular Momentum

The **angular momentum** $\vec{\mathbf{L}}$ of a particle relative to an axis through the origin is defined by the cross product of the object's position vector $\vec{\mathbf{r}}$ and its instantaneous linear momentum $\vec{\mathbf{p}}$.

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$

Analogous to Newton's 2nd law,

$$\sum \vec{\boldsymbol{\tau}} = \frac{d\vec{\mathbf{L}}}{dt}$$

Since $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, the magnitude of $\vec{\mathbf{L}}$ is

$$L = mvr \sin\phi$$

where r is the distance of the particle from the origin, and ϕ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{p}}$.

L is zero when $\vec{\mathbf{r}}$ is parallel to $\vec{\mathbf{p}}$, whereas $L = mvr$ when the vectors are perpendicular to each other.

Furthermore, for a rigid object rotating in the xy-plane about the z-axis with an angular speed ω, its angular momentum is

$$L_z = \sum_i L_i = \left(\sum_i m_i {r_i}^2\right)\omega = \boxed{I\omega}$$

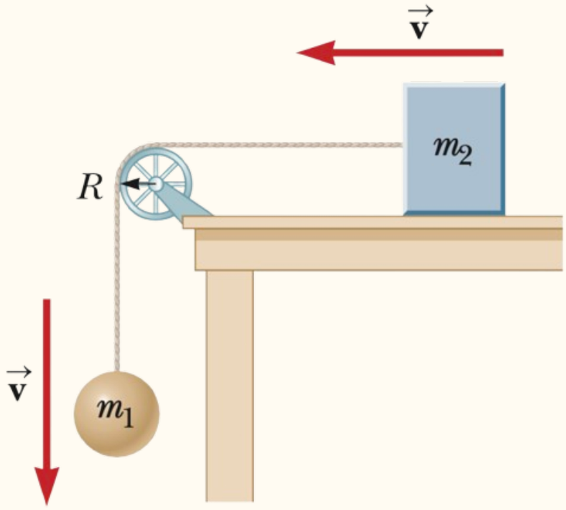
A2.4.1 System of Particles

For a system that rotates about an axis, if there is a net external torque acting on the system, *the rate of change of angular momentum is equal to the torque.*

$$\boxed{\sum \vec{\boldsymbol{\tau}} = \frac{d\vec{\mathbf{L}}_{\text{tot}}}{dt} \iff \Delta\vec{\mathbf{L}}_{\text{tot}} = \int \sum \vec{\boldsymbol{\tau}}\, dt}$$

Example A2.5

A sphere of mass m_1 and a block of mass m_2 are connected by a light cord that passes over a pulley as shown.



The radius of the pulley is R, and the mass of the thin rim is M. The spokes of the pulley have negligible mass. The block slides on a frictionless, horizontal surface.

Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.

Since all the objects move with equal speed, the total angular momentum of the sphere, block, and pulley is

$$L = (m_1 + m_2 + M)vR$$

Substituting the expression for total external torque,

$$\begin{aligned}
\sum \tau = \frac{dL}{dt} &\implies m_1 gR = \frac{d}{dt}[(m_1 + m_2 + M)vR] \\
&\implies m_1 gR = (m_1 + m_2 + M)R\frac{dv}{dt} \\
&\implies a = \boxed{\frac{m_1 g}{m_1 + m_2 + M}}
\end{aligned}$$

A2.4.2 Conservation of Angular Momentum

Definition A2.5:

For an *isolated system*, the **conservation of angular momentum** states that the total angular momentum of a system is constant in both magnitude and direction if the net external torque acting on the system is zero.

By this principle,

$$\begin{aligned}
\sum \vec{\tau} = 0 &\implies \frac{d\vec{\mathbf{L}}_{\text{tot}}}{dt} = 0 \\
&\implies \boxed{\Delta \vec{\mathbf{L}}_{\text{tot}} = 0}
\end{aligned}$$

Hence, using the formula $L = I\omega$,

$$\vec{\mathbf{L}}_i = \vec{\mathbf{L}}_f \implies I_i\omega_i = I_f\omega_f = \text{constant}$$

A2.5 Rotational Kinetic Energy

Definition A2.6: Rotational Kinetic Energy

The **rotational kinetic energy** of a system of particles of moment of inertia I and rotating with angular speed ω is

$$K_R = \frac{1}{2}I\omega^2$$

This is because it is the *sum of all the kinetic energies* of each individual particle.

$$K_R = \sum_i K_i = \sum_i \frac{1}{2}m_i{v_i}^2 = \frac{1}{2}\left(\sum_i m_i{r_i}^2\right)\omega^2$$

A2.5.1 Rotational Work

Definition A2.7: Rotational Work

The **work done** by a force causing a rotational torque τ to move through an angle from θ_i to θ_f is

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$$

Proof. To derive this, when an external force $\vec{\mathbf{F}}$ is applied at a point on a rigid object, causing it to rotate through an infinitesimal distance $ds = r\,d\theta$, we see that

$$dW = \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = (F\sin\phi)r\,d\theta = \tau\,d\theta$$

Therefore,

$$\boxed{W = \int_{\theta_i}^{\theta_f} \tau \, d\theta}$$

□

A2.5.2 Work-Energy Theorem

Definition A2.8: Rotational Work-Energy Theorem

The **work-energy theorem** also holds **for rotational dynamics**, which states that the *net work done* by external forces in *rotating a symmetrical rigid object* about a fixed axis equals to the *change in the object's rotational energy.*

Proof. Examining net torque,

$$\sum \tau = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\frac{d\omega}{d\theta}\omega$$

Note that from above, $dW = \sum \tau \, d\theta$. Hence,

$$\sum \tau \, d\theta = dW = I\omega \, d\omega \implies W = \int_{\omega_i}^{\omega_f} I\omega \, d\omega = \frac{1}{2}I{\omega_f}^2 - \frac{1}{2}I{\omega_i}^2$$

□

A2.5.3 Power

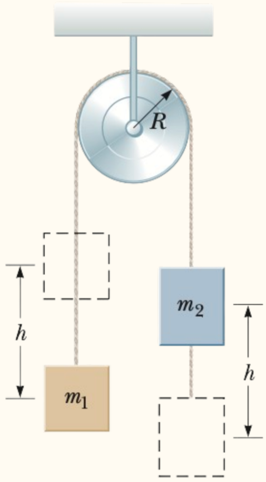
The **power** delivered by such a force is

$$P = \frac{dW}{dt} = \boxed{\tau\omega}$$

$$\langle P \rangle = \frac{\Delta W}{\Delta t}$$

Example A2.6

Two blocks having different masses m_1 and m_2 are connected by a string passing over a pulley.



The pulley has a radius R and moment of inertia I about its axis of rotation. The system is released from rest.

Using this information, find

1. the translational speeds of the blocks after block 2 descends through a distance h, and
2. the angular speed of the pulley at this time.

Using the isolated system model, where v_f is the final translational speed of both blocks, and ω_f is the final angular speed of the pulley,

$$\Delta K = -\Delta U \implies \frac{1}{2}m_1{v_f}^2 + \frac{1}{2}m_2{v_f}^2 + \frac{1}{2}I{\omega_f}^2 = -(m_1gh - m_2gh)$$

Substituting $v = R\omega$,

$$\frac{1}{2}m_1{v_f}^2 + \frac{1}{2}m_2{v_f}^2 + \frac{1}{2}I\frac{{v_f}^2}{R^2} = m_2gh - m_1gh \implies \frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right){v_f}^2 = (m_2 - m_1)gh$$

$$\implies v_f = \boxed{\sqrt{\frac{2(m_2 - m_1)gh}{m_1 + m_2 + I/R^2}}}$$

$$\implies \omega_f = \boxed{\frac{1}{R}\sqrt{\frac{2(m_2 - m_1)gh}{m_1 + m_2 + I/R^2}}}$$

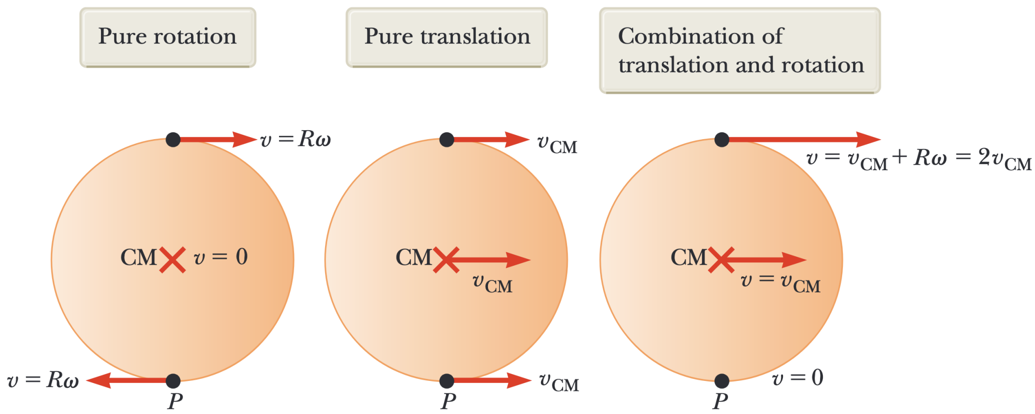
A2.6 Rolling Motion

For a uniform cylinder of radius R undergoing **pure rolling motion** (i.e. no slipping), its centre of mass will always follow a straight-line trajectory.

$$v_{\text{CM}} = R\omega_{\text{CM}} \implies a_{\text{CM}} = R\alpha_{\text{CM}}$$

This holds only for cylinders and spheres undergoing pure rolling motion. More practically, **rolling friction** may cause mechanical energy to transform to internal energy. This rolling friction needs to be considered in most questions.

It is important to consider the combination between the rotational motion of the object and the translational motion of the object.



When we consider the combination, we can model the rolling motion of a body as being equivalent to pure rotation about a fixed pivot point P (where the body touches the ground). As such, the *total kinetic energy* of the rolling body is given as

$$K = \frac{1}{2}I_p\omega^2$$

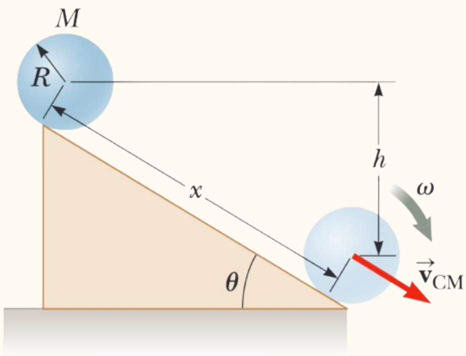
Applying the parallel-axis theorem and the pure rolling equation $v_{\text{CM}} = R\omega_{\text{CM}}$,

$$K = \frac{1}{2}I_p\omega^2 \implies K = \frac{1}{2}I_{\text{CM}}\omega^2 + \frac{1}{2}(MR^2)\omega^2$$

$$\implies \boxed{K = \frac{1}{2}I_{\text{CM}}\omega^2 + \frac{1}{2}M{v_{\text{CM}}}^2}$$

Example A2.7

A uniform cylinder of radius R and mass M rolls from rest without slipping down an inclined plane which makes an angle θ with the horizontal. The center of mass of the cylinder has dropped a vertical distance h when it reaches the bottom of the inclined plane.



Ignoring rolling friction, what is the speed of the center of the mass of the cylinder when it reaches the bottom of the inclined in terms of h and the acceleration of free fall g only?

Since the cylinder undergoes pure rolling motion, the equation $v_{CM} = R\omega_{CM}$ applies. The final kinetic energy of the cylinder is

$$K_f = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}M{v_{CM}}^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 + \frac{1}{2}Mv^2$$

Due to the conservation of energy,

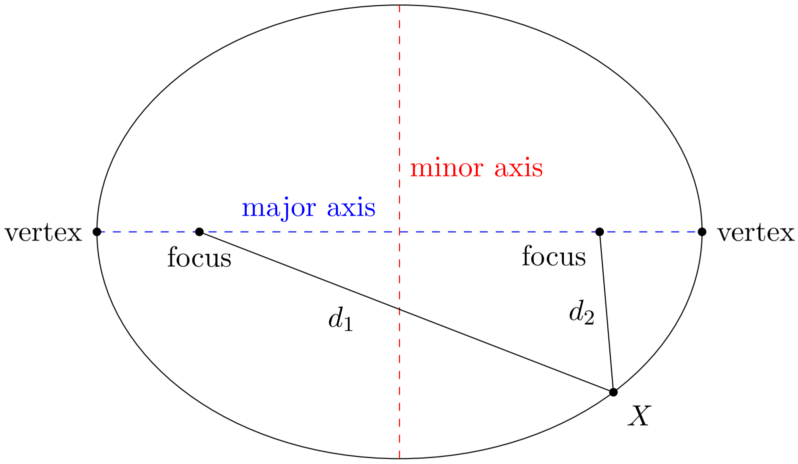
$$\begin{aligned}
\Delta K + \Delta U = 0 \implies Mgh &= \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 + \frac{1}{2}Mv^2 \\
\implies gh &= \frac{1}{4}v^2 + \frac{1}{2}v^2 \\
\implies v^2 &= \frac{4}{3}gh \\
\implies & \boxed{v = \sqrt{\frac{4}{3}gh}}
\end{aligned}$$

A3 Planetary & Satellite Motion

A3.1 Elliptical Orbits

A3.1.1 Ellipse

An ellipse is mathematically defined by choosing two points, each called a **focus**, and drawing a curve through which the sum of distances from each focus d_1 and d_2 is a constant.



The longest distance through the center between points on the ellipse is the **major axis** (with the line from one center to a vertex being a **semi-major axis**), whereas the shortest distance through the center is a **minor axis**.

The **eccentricity** of an ellipse is defined as

$$e = \frac{c}{a}, \quad e \in (0, 1)$$

where c is the distance of each focus from the centre, and a is the length of the semi-major axis.

Eccentricity describes shape. For a circle, the focus is at the origin, hence $c = 0 \implies e = 0$. The *longer and thinner* an ellipse is, the *greater* e is.

A3.1.2 Periapsis & Apoapsis

The **periapsis** is the point of closest approach, and the **apoapsis** is the point of furthest approach. The periapsis and apoapsis are vertices on the elliptical orbit, the arrangement depending which focus the body is orbiting around.

The **distance** of the periapsis and apoapsis are denoted as r_p and r_a respectively. The length of the semi-major axis is a, whereas the length of the minor axis is b.

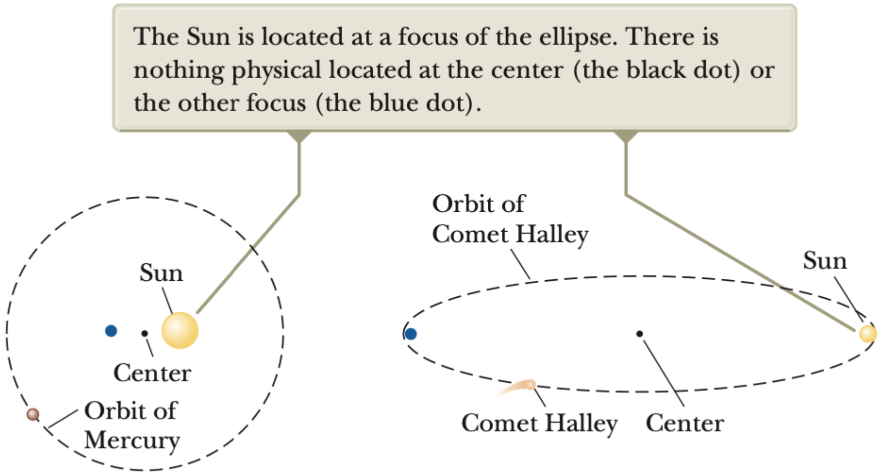
$$a = \frac{1}{2}(r_p + r_a)$$

A3.2 Kepler's Laws

A3.2.1 Kepler's First Law

Definition A3.1: Kepler's First Law

Kepler's first law states that the orbit of a planet is an ellipse with the Sun at one of the two foci.



The **aphelion** is the point where the planet is farthest away from the Sun, and the **perihelion** is the point where the planet is nearest to the Sun.

A3.2.2 Kepler's Second Law

Definition A3.2: Kepler's Second Law

Kepler's second law states that the radius vector drawn from the Sun to a moving planet *sweeps out equal areas in equal time intervals.*

Proof. This is a result of the conservation of angular momentum. For a planet of mass M moving around the Sun with *no net torque* (since gravitational force $\vec{\mathbf{F}}$ is parallel to $\vec{\mathbf{r}}$),

$$\begin{aligned} \Delta\vec{\mathbf{L}} = 0 &\implies \vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = \text{constant} \\ &\implies L = M|\vec{\mathbf{r}} \times \vec{\mathbf{v}}| = \text{constant} \end{aligned}$$

Considering a geometric interpretation, in a time interval dt, the radius vector $\vec{\mathbf{r}}$ sweeps out a triangle of area dA, given as

$$dA = \frac{1}{2}|\vec{\mathbf{r}} \times d\vec{\mathbf{r}}| = \frac{1}{2}|\vec{\mathbf{r}} \times \vec{\mathbf{v}}|\, dt$$

Substituting the earlier expression,

$$\frac{dA}{dt} = \frac{L}{2M} = \text{constant} \implies \text{equal areas in equal time intervals}$$

□

A3.2.3 Kepler's Third Law

Definition A3.3: Kepler's Third Law

Kepler's third law states that the ratio of the *square of a planet's period of revolution* to the *cube of the semi-major axis* of its orbit around the Sun is constant.

Proof. We begin with Kepler's second law. For a orbital period of T, the total area swept out in one orbit is

$$\frac{dA}{dt} = \frac{L}{2m} \implies A = \frac{L}{2m}T$$

An ellipse of semi-major and semi-minor axes a and b has an area described by

$$A = \pi ab = \pi a^2\sqrt{1-e^2}$$

Substituting this area,

$$\begin{aligned}\pi a^2\sqrt{1-e^2} = \frac{L}{2m}T &\implies T = \frac{m}{L}2\pi a^2\sqrt{1-e^2} \\ &\implies T^2 = \frac{m^2}{L^2}4\pi^2 a^4(1-e^2)\end{aligned}$$

Importantly, we can express the semi-latus rectum ℓ of an ellipse using the equation $\ell = a(1-e^2)$. Using the expression $\ell = L^2/(GMm^2)$,

$$\begin{aligned}T^2 &= \frac{1}{GM\ell}\cdot 4\pi^2 a^3\ell = \frac{4\pi^2}{GM}a^3 \\ &\implies \text{ǫ}xedT^2 \propto a^3\end{aligned}$$

Thus, we conclude that the ratio of the square of the orbital period to the cube of the semi-major axis is always constant, with the constant of proportionality being $4\pi^2/(GM)$. □

A3.3 Newton's Shell Theorem

Definition A3.4: Shell Theorem

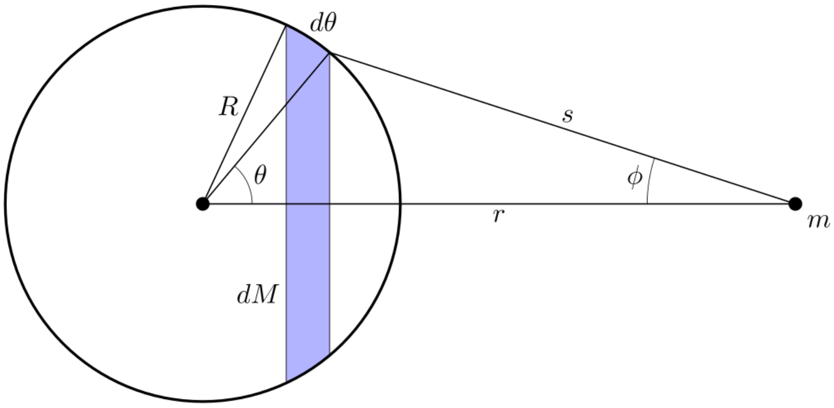
The **shell theorem** comprises of two colloraries, stating that

1. a spherically symmetric body affects an external point mass as though all the mass of the spherical body were *concentrated at its center*;
2. if the body is a spherically symmetric shell, *no net gravitational force is exerted* by the shell *on any object inside*, regardless of the object's location within the shell.

Consider a point mass m at a distance r from the centre of a spherical shell of mass M and radius R.

Consider a thin ring of mass corresponding to an angle θ between r and any point on the shell. Let s denote the distance of m and the ring.

Letting $\mu = \dfrac{M}{4\pi R^2}$ be the surface mass density of the shell, we can find the mass of a thin ring dm



as shown.

$$\begin{aligned} dM &= \mu(2\pi R\sin\theta)(R\,d\theta) \\ &= \left(\frac{M}{4\pi R^2}\right)(2\pi R^2\sin\theta)\,d\theta \\ &= \left(\frac{M}{2}\sin\theta\right)\,d\theta \end{aligned}$$

By cosine rule,

$$s^2 = R^2 + r^2 - 2Rr\cos\theta$$

Differentiating with respect to θ, we obtain

$$2s\left(\frac{ds}{d\theta}\right) = 2Rr\sin\theta \implies \sin\theta = \frac{s}{Rr}\left(\frac{ds}{d\theta}\right)$$

The gravitational potential energy between the ring of mass dM and mass m is

$$\begin{aligned} dU &= -\frac{Gm}{s}\,dM \\ &= -\frac{Gm}{s}\left(\frac{M}{2}\sin\theta\right)\,d\theta \\ &= -\frac{Gm}{s}\left(\frac{M}{2}\cdot\frac{s}{Rr}\left(\frac{ds}{d\theta}\right)\right)\,d\theta \\ &= -\frac{GMm}{2Rr}\,ds \end{aligned}$$

To prove the **first collorary**, we consider the region outside the shell, $r > R$.

$$U = \int_{r-R}^{r+R} -\frac{GMm}{2Rr}\,ds = -\frac{GMm}{2Rr}[s]_{r-R}^{r+R} = \boxed{-\frac{GMm}{r}}$$

Therefore,

$$F = -\frac{dU}{dr} = -\frac{GMm}{r^2}$$

To prove the **second collorary**, we consider the region inside the shell, $r < R$.

$$U = \int_{R-r}^{r+R} -\frac{GMm}{2Rr}\,ds = -\frac{GMm}{2Rr}[s]_{R-r}^{r+R} = \boxed{-\frac{GMm}{R}}$$

Therefore,

$$F = -\frac{dU}{dr} = 0$$

A3.4 Effective Radial Potential

Definition A3.5: Effective Radial Potential

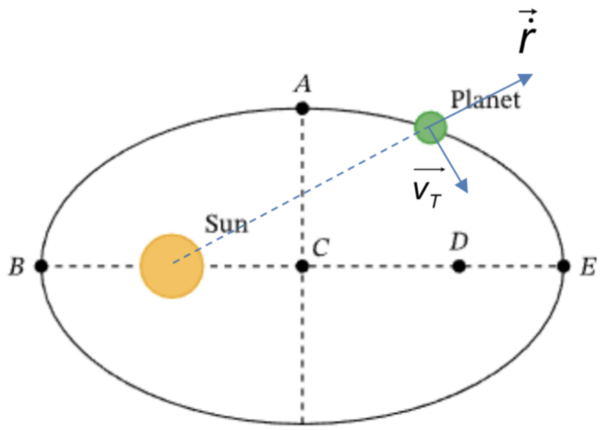
The **effective radial potential energy** of a mass m is defined as the sum of the r-dependent terms of its energy.

$$U_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

where the first term is the **transverse kinetic energy**, and the second term is the **gravitational potential energy**.

The effective radial potential of the mass m may be used to determine the *orbits* of planets, and perform semi-classical atomic calculations.

A3.4.1 Derivation



The velocity of a mass m orbiting a larger mass M can be resolved into the radial component v_r and the tranverse component v_t.

Thus, the kinetic energy of the mass is given as

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m({v_r}^2 + {v_t}^2)$$

From the angular momentum of the mass,

$$\begin{aligned} L = m\vec{\mathbf{v}} \times \vec{\mathbf{r}} = mv_t r &\implies v_t = \frac{L}{mr} \\ &\implies \frac{1}{2}m{v_t}^2 = \frac{L^2}{2mr^2} \end{aligned}$$

Considering the Hamiltonian of the system,

$$E = K + U = \frac{1}{2}m{v_r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

Therefore, taking the terms that depend on r, we obtain the effective radial potential.

$$U_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

A3.4.2 Turning Points

A **turning point** is a finite radial value $r = r^*$ such that

$$\begin{aligned} U_{\text{eff}} = E &\implies \frac{1}{2}m{v_r}^2 = 0 \\ &\implies v_r = 0 \end{aligned}$$

This implies that a turning point exists in a path of mass m where it has no radial component to its velocity.

Example A3.1

Determine the roots of $U_{\text{eff}} = E$. Hence, or otherwise, show that the length of the planet's semi-major axis is given by

$$a = -\frac{GMm}{2(E)}$$

From $U_{\text{eff}} = \sum E$,

$$\frac{L^2}{2m}\left(\frac{1}{r}\right)^2 - GMm\left(\frac{1}{r}\right) - E = 0$$

Using the quadratic formula,

$$\begin{aligned} \frac{1}{r} &= \frac{GMm \pm \sqrt{(GMm)^2 - 4(L^2/2m)(-E)}}{L^2/m} \\ &= \frac{1 \pm \sqrt{1 + 2L^2(E)/m(GMm)^2}}{L^2/(GMm)m} \\ \implies \frac{L^2}{GMm^2}\frac{1}{r} &= 1 \pm \sqrt{1 + \frac{2L^2(E)}{m(GMm)^2}} \end{aligned}$$

Finding the length of the semi-major axis, we employ Vieta's formula. Recall that r_p and r_a represent the distances of the periapsis and apoapsis respectively.

$$\frac{1}{r_a} + \frac{1}{r_p} = \frac{r_a + r_p}{r_a r_p} = \frac{GMm}{L^2/2m}$$

$$r_a r_p = \left(\frac{1}{r_a} \cdot \frac{1}{r_p}\right)^{-1} = -\frac{L^2/2m}{E}$$

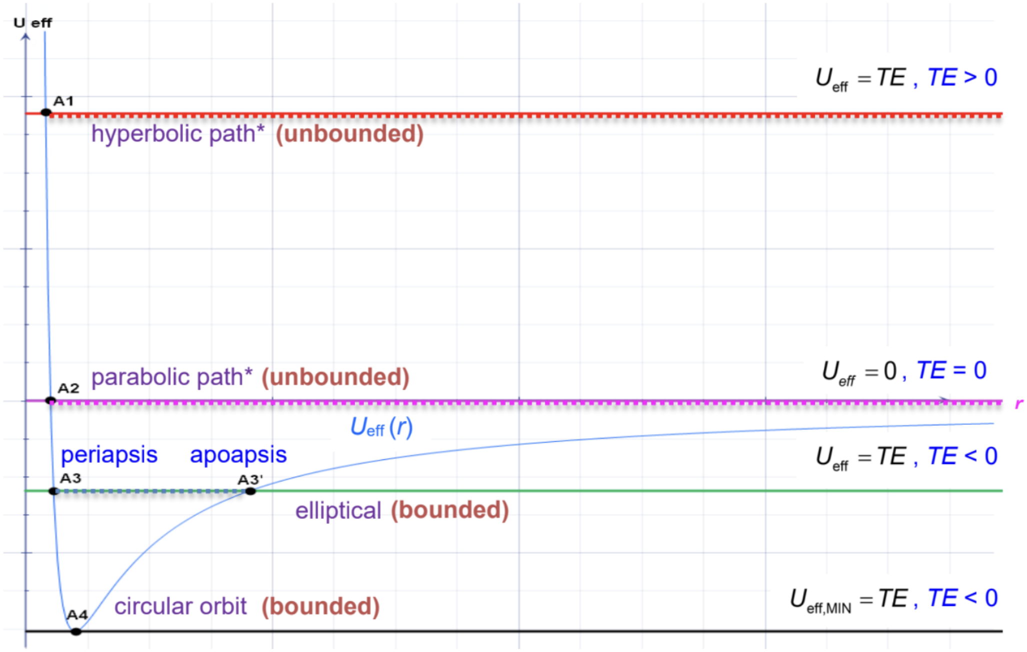
Since $a = \frac{1}{2}(r_a + r_p)$,

$$a = \frac{1}{2} \cdot \frac{GMm}{L^2/2m} \cdot -\frac{L^2/2m}{E}$$

$$a = \boxed{-\frac{GMm}{2E}}$$

A3.4.3 Bounded States

The graph below shows the relationship of U_{eff} against r.



Circular Orbits

The lowest energy state $U_{\text{eff, min}}$ corresponds to the *minimum* of the effective radial potential.

Setting the derivative of U_{eff} to zero,

$$\frac{dU_{\text{eff}}}{dt} = -\frac{L^2}{mr^3} + \frac{GMm}{r^2} = 0r = \frac{L^2}{GMm^2}$$

For a circular orbit, the distance between m and M does not change.

$$v_t = \frac{L}{mr} = \frac{GMm}{mv_t r} = \sqrt{\frac{GM}{r}}$$

The speed calculated above is the speed *required for a circular orbit*. Let this speed be v_c.

Elliptical Orbits

Recall that the **escape velocity** is given as

$$v_e = \sqrt{\frac{2GM}{r}}$$

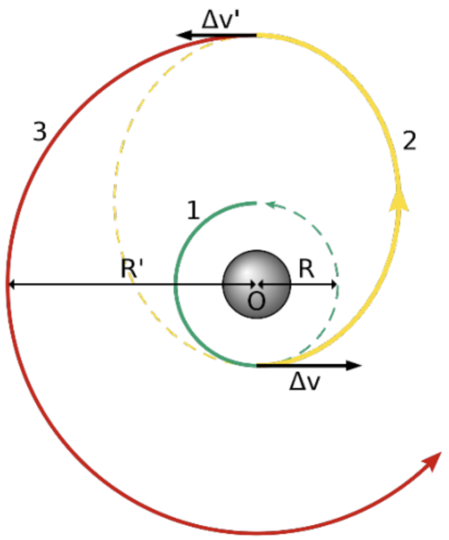
The speed of the elliptical orbit can take on two states, depending on whether it has surpassed the speed required for a circular orbit.

$$0 < v_t < v_c \text{ or } v_c < v_t < v_e$$

A3.5 Hohmann Orbital Transfer

A **Hohmann orbital transfer** is a two-impulse elliptical transfer between two co-planar circular orbits.

The transfer itself consists of a periapsis at the inner orbit, before moving into an elliptical orbit, and ending off with an apoapsis at the outer orbit.



One assumption is that *velocity change occurs instantaneously* for the spacecraft. In the following, we examine the efficiency of the Hohmann orbital transfer.

If $\vec{\mathbf{v}}_i$ is the velocity of the initial orbit and $\vec{\mathbf{v}}_f$ is the velocity of the final orbit, the change in velocity $\Delta\vec{\mathbf{v}}$ required to perform the orbital transfer is

$$\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i$$

Using the law of cosines,

$${v_f}^2 = {v_i}^2 + \Delta v^2 - 2(v_i)(\Delta v)\sin\beta$$

where β is the angle between $\vec{\mathbf{v}}_i$ and $\Delta\vec{\mathbf{v}}$.

The **specific energy charge** (energy change per unit mass) is given by

$$\frac{\Delta E}{m} = \frac{1}{2}\Delta v^2 + (v_i)(\Delta v)\cos\beta$$

Note that this specific energy change is greatest when $\beta = 0$ and v_i is at its maximum. In other words, the most energy-efficient directions of impulse would be collinear with the tangential velocity and apply it at the elliptical orbit's periapsis, where speed is greatest.

Example A3.2

A communication satellite was carried by the Space Shuttle into low earth orbit at an altitude of 322 km and is to be transferred to a geostationary orbit at an alitude of 35 860 km using a Hohmann transfer.

Determine the magnitude of the total change in velocity after the Hohmann transfer is complete given the radius of Earth is 6 378 km.

For the low earth orbit,

$$\begin{aligned} v_1 = \sqrt{\frac{GM}{r_1}} &= R_{\text{earth}}\sqrt{\frac{g}{r_1}} \\ &= 7718 \text{ m s}^{-1} \end{aligned}$$

For the geostationary orbit,

$$\begin{aligned} v_2 = \sqrt{\frac{GM}{r_2}} &= R_{\text{earth}}\sqrt{\frac{g}{r_2}} \\ &= 3073 \text{ m s}^{-1} \end{aligned}$$

For the Hohmann transfer orbit, using the conservation of energy, at the periapsis,

$$\begin{aligned} K + U = \frac{-GMm}{2a} &\implies \frac{1}{2}m{v_p}^2 - \frac{GMm}{r_1} = -\frac{GMm}{(r_1 + r_2)} \\ &\implies -\frac{gR_{\text{earth}}^2}{r_1 + r_2} = \frac{1}{2}{v_p}^2 - \frac{gR_{\text{earth}}^2}{r_1} \\ &\implies \frac{1}{2}{v_p}^2 = gR_{\text{earth}}^2\left(\frac{r_2}{r_1(r_1 + r_2)}\right) \\ &\implies v_p = R_{\text{earth}}\sqrt{2g\left(\frac{r_2}{r_1(r_1 + r_2)}\right)} \\ &\implies v_p = 10140 \text{ m s}^{-1} \end{aligned}$$

Similarly, at the apoapsis,

$$\begin{aligned} K + U = \frac{-GMm}{2a} &\implies \frac{1}{2}m{v_p}^2 - \frac{GMm}{r_2} = -\frac{GMm}{(r_1 + r_2)} \\ &\implies v_a = R_{\text{earth}}\sqrt{2g\left(\frac{r_1}{r_2(r_1 + r_2)}\right)} \\ &\implies v_a = 1608 \text{ m s}^{-1} \end{aligned}$$

Therefore, the magnitude of the total change in velocity

$$\Delta v = (v_p - v_1) + (v_2 - v_a) = (10140 - 7718) + (3073 - 1608) \implies \boxed{\Delta v = 3890 \text{ m s}^{-1}}$$

B1 Electric & Magnetic Fields

B1.1 Properties of Electric Charges

There are three main properties of electric charges.

1. Charge comes in two varieties, termed **positive** (protons) and **negative** (electrons). Charges of the same sign **repel** one another and charges with opposite signs **attract** one another.
2. Charge is **always conserved** in an isolated system. Charge is neither created nor destroyed, but rather transferred from one object to another.
   - The **global** conservation of charge states that the total charge of the universe is fixed for all time.
   - The **local** conservation of charge states that for a charge to be transferred from one location to another, it must have passed along some continuous path (i.e. charges cannot suddenly teleport)
3. Charge is **quantised**, coming only in discrete lumps.
   - For an electric charge q, we can write $q = \pm Ne,\ N \in \mathbb{Z}$.
   - The electron has a charge $-e$ and the proton has a charge $+e$. The neutron carries no charge.

B1.1.1 Conductors

Electrical conductors are materials in which some of the electrons are free electrons that are not bound to atoms and can move relatively freely through the material. This includes copper, aluminium, and silver.

Likewise, **electrical insulators** are materials in which all electrons are bound to atoms and cannot move freely through the material. This includes glass, rubber, and dry wood.

Semiconductors are between insulators and conductors, with its electrical properties able to be changed over many orders of magnitude by the addition of controlled amounts of certain atoms to the materials. This includes silicon and germanium.

B1.1.2 Charging by Induction

Conductors are charged by a process known as **induction**.

Consider a neutral conducting sphere insulated from the ground. Since the charge is zero, there are an *equal numhber of electrons and protons* in the sphere.

- When a *negatively-charged* rubber rod is brought near the sphere, electrons in the sphere are repulsed and move to the opposite side of the sphere.
- This migration leaves the side of the sphere near the rod with an *effective positive charge* (purely due to the absence of electrons, for *protons do not move*).
- If the sphere has a *ground*, the electrons in the conductor are so strongly repelled that they move into the Earth.

- In this scenario, positive charges are *induced* on the sphere. For charging by induction, the position of the grounding wire does not matter — the induced charges will always remain on the sphere.

Charging an object by induction requires no contact. This is in contrast to charging by rubbing (by conduction).

B1.2 Electric Field

To begin, it may be important to highlight the *principle of superposition*.

$$\vec{\mathbf{F}}_{\text{net}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots + \vec{\mathbf{F}}_n$$

We consider the case of **electrostatics** where all source charges are stationary (but the test charge may be moving). We use the term **point charge** to refer to a charged particle of negligible size.

B1.2.1 Coulomb's Law

Definition B1.1: Coulomb's Law (H3)

Coulomb's law describes the **electric force** between two charges.

$$\vec{\mathbf{F}} = \frac{1}{4\pi\epsilon_o}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$$

where

- $\vec{\mathbf{F}}$ is the electric force (N)
- q_1 is the charge of the first point charge (C)
- q_2 is the charge of the second point charge (C)
- r is the separation between the two charges (m)
- $\hat{\mathbf{r}}$ is the separation unit vector from q_1 to q_2, purely to provide direction
  - $\vec{\mathbf{r}} = \vec{\mathbf{x}} - \vec{\mathbf{x}}'$, where $\vec{\mathbf{x}}$ is the location of q_1 and $\vec{\mathbf{x}}'$ is the location of q_2
- ϵ_0 is the permittivity of free space, with a value of $8.85 \times 10^{-12}\ \mathrm{C^2\,kg^{-1}\,m^{-3}\,s^2}$

If $q_1 q_2$ is positive, the electric force on one particle is directed away from the other particle. If the product is negative, the electric force on both particles point towards each other.

B1.2.2 Electric Field Strength

If we have *several* point charges $q_1, q_2, \cdots, q_n$ at distances $r_1, r_2, \cdots, r_n$ from a source charge Q, the total force on Q is

$$\begin{aligned}
\sum \vec{\mathbf{F}} &= \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots + \vec{\mathbf{F}}_n \\
&= \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{{r_1}^2}\hat{\mathbf{r}}_\mathbf{1} + \frac{q_2 Q}{{r_2}^2}\hat{\mathbf{r}}_\mathbf{2} + \cdots + \frac{q_n Q}{{r_n}^2}\hat{\mathbf{r}}_\mathbf{3}\right) \\
&= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{{r_1}^2}\hat{\mathbf{r}}_\mathbf{1} + \frac{q_2}{{r_2}^2}\hat{\mathbf{r}}_\mathbf{2} + \cdots + \frac{q_n}{{r_n}^2}\hat{\mathbf{r}}_\mathbf{n}\right)
\end{aligned}$$

This can be expressed as

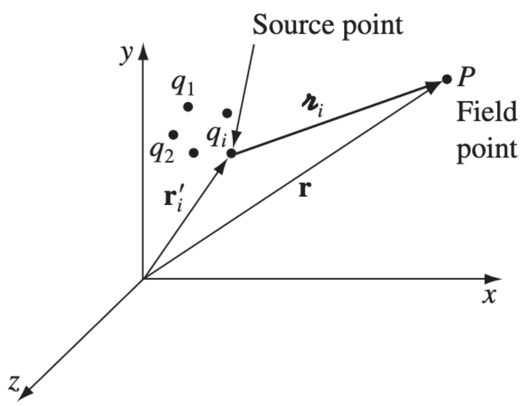
$$\vec{\mathbf{F}}_{\text{net}} = q\vec{\mathbf{E}}$$

Definition B1.2: Electric Field Strength (H3)

The electric field strength $\vec{\mathbf{E}}$ at a point in the field is defined as the electric force per unit positive charge experienced by a charge placed at that point.

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{{r_i}^2}\hat{\mathbf{r}}_{\mathbf{i}}$$

$\vec{\mathbf{E}}$ is a function of the position vector $\vec{\mathbf{r}}$, which characterising the distance between the source point q_i and a field point Q. This is because the separation vectors $\vec{\mathbf{r}}_{\mathbf{i}}$ depend on the location of the **field point** Q.



Physically, $\vec{\mathbf{E}}(\vec{\mathbf{r}})$ is the force per unit charge that would be exerted on a positive test charge, if you were to place one at P.

B1.2.3 Electric Field Lines

Electric field patterns are represented by **electric field lines**, related to the electric field in a region of space in the following manner.

1. The *electric field vector* $\vec{\mathbf{E}}$ is tangential and parallel to the electric field line at each point. The direction of the line is that of the *force on a positive charge* placed in the field.

2. The number of lines per unit area is proportional to the magnitude of the electric field in that region — field lines are close together for stronger fields, and far apart for weaker fields.

Field lines begin on positive charges and end on negative charges — they cannot terminate in midair.

For two charges of equal magnitude, they cancel out. The number of field lines leaving the positive charge must be equal to the number of field lines terminating at the negative charge.

This is similar to what is described in the H2 syllabus.

B1.2.4 Motion of Charged Particles

When a particle of charge q and mass m is placed in an electric field $\vec{\mathbf{E}}$, the electric force exerted on the charge is $q\vec{\mathbf{E}}$.

Given that this force is the net force,

$$\vec{\mathbf{F}}_{\text{net}} = q\vec{\mathbf{E}} = m\vec{\mathbf{a}}$$

therefore giving the particle's acceleration to be

$$\vec{\mathbf{a}} = \frac{q\vec{\mathbf{E}}}{m}$$

B1.2.5 Continuous Charge Distribution

The previous formula for electric field strength holds true *only if* the source of the field is a collection of **discrete point chrages**.

If the charge is distributed continuously over some region, then the sum becomes an integral. This integration is a vector operation and must be treated appropriately.

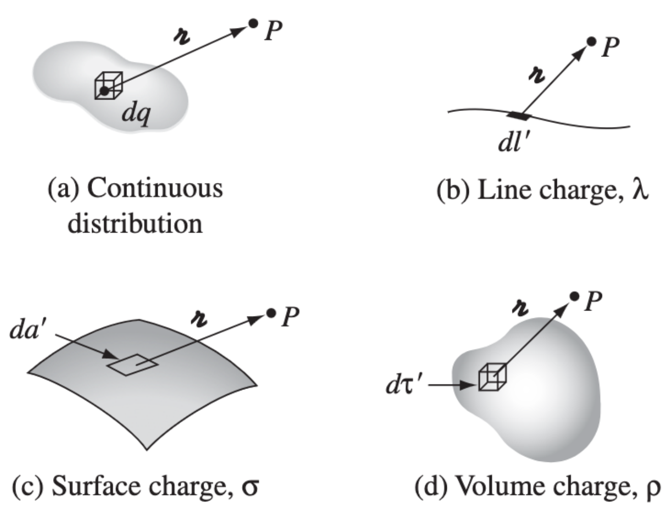
$$\vec{\mathbf{E}}(\vec{\mathbf{x}}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{r}\hat{\mathbf{r}}\, dq$$

This calculation can be extended for a line charge, area charge, and volume charge. We can use the concept of **charge density**. If a charge Q is uniformly distributed along a line of length l / area A / volume V, the linear / surface / volume charge density λ/ σ / ρ (C/m or C/m^2 or C/m^3) is

$$\lambda = \frac{Q}{l} \qquad \sigma = \frac{Q}{A} \qquad \rho = \frac{Q}{V}$$

If the charge is non-uniformly distributed over a line, surface, or volume, the amounts of charge dq in a small volume is equivalent to the following infinitesimal values.

$$dq \to \lambda\, dl \sim \sigma\, dA \sim \rho\, dV$$



Consequently, specifically, for a line charge,

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\vec{\mathbf{x}}')}{r^2}\hat{\mathbf{r}}\,dl$$

For a surface charge,

$$\vec{\mathbf{E}}(\vec{\mathbf{x}}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{\mathbf{x}}')}{r^2}\hat{\mathbf{r}}\,dA$$

And, rather importantly, for a volume charge,

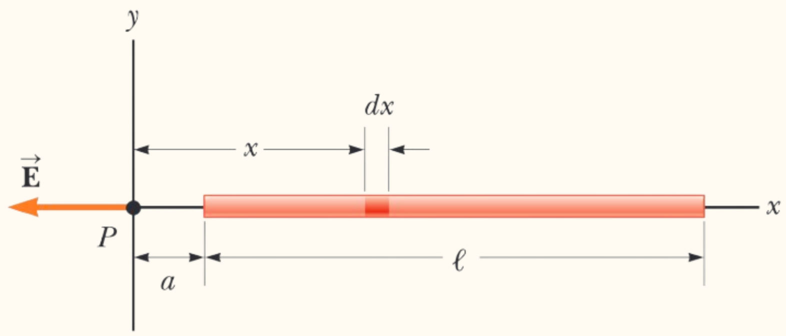
$$\vec{\mathbf{E}}(\vec{\mathbf{x}}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\mathbf{x}}')}{r^2}\hat{\mathbf{r}}\,dV$$

where $\vec{\mathbf{r}}$ is the vector from dq (and naturally from dl, dA and dV) to the field point $\mathbf{x}$.

Uniformly Charged Rod

Example B1.1

A rod of length l has a uniform positive charge per unit length λ and a total charge Q. We wish to calculate the electric field at a point P that is located along the long axis of the rod and a distance a from one end.



Consider an infinitesimal element of the rod with length dx. The charge on this segment is dq, which can be expanded to form $\lambda\ dx$. Thus, the electric field contribution from this segment is given by

$$d\vec{\mathbf{E}} = -\frac{1}{4\pi\epsilon_0}\frac{dq}{x^2}\hat{\mathbf{x}} = -\frac{1}{4\pi\epsilon_0}\frac{\lambda\,dx}{x^2}\hat{\mathbf{x}}$$

(The negative sign in this equation is due to the fact that the electric field points rightward, where we take leftwards to be positive).

Integrating, we obtain the total electric field

$$\begin{aligned}\vec{\mathbf{E}} = \int d\vec{\mathbf{E}} &= -\frac{1}{4\pi\epsilon_0}\int_a^{a+l}\frac{\lambda dx}{x^2}\hat{\mathbf{x}} \\ &= -\frac{\lambda}{4\pi\epsilon_0}\left[-\frac{1}{x}\right]_a^{a+l}\hat{\mathbf{x}}\end{aligned}$$

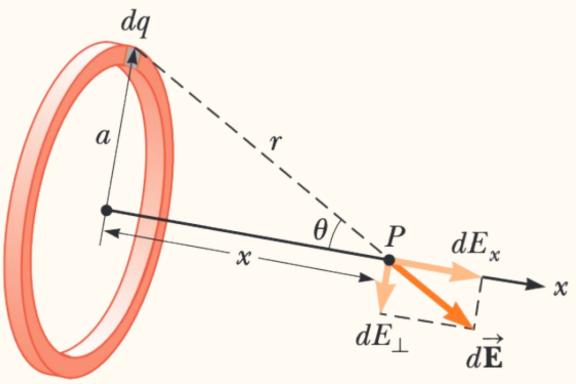
Since $\lambda = \frac{Q}{l}$,

$$\vec{\mathbf{E}} = -\frac{1}{4\pi\epsilon_0}\frac{Q}{l}\left(\frac{1}{a} - \frac{1}{a+l}\right)\hat{\mathbf{x}} = \boxed{-\frac{1}{4\pi\epsilon_0}\frac{Q}{a(a+l)}\hat{\mathbf{x}}}$$

Uniformly Charged Ring

Example B1.2

A ring of radius a carries a uniformly distributed positive total charge Q. Calculate the electric field due to the ring at a point P lying a distance x from its center along the central axis perpendicular to the plane of the ring.



Consider an infinitesimal element of the ring with length dx. We notice that for each element, there is another element directly opposite to it on the ring. With reference to P, the vertical component of each element will be cancelled out, leading to the overall electric field solely possessing a horizontal component.

The charge on this segment is dq.

Thus, the electric field contribution from the segment is

$$d\vec{\mathbf{E}} = dE_x\hat{\mathbf{x}} = |d\vec{\mathbf{E}}|\cos\theta\,\hat{\mathbf{x}} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\cos\theta\,\hat{\mathbf{x}}$$

Integrating, we obtain the total electric field.

$$\begin{aligned}\vec{\mathbf{E}} = \int d\vec{\mathbf{E}} &= \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r^2}\cos\theta\,\hat{\mathbf{x}} \\ &= \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\sqrt{x^2+a^2}^2}\frac{x}{\sqrt{x^2+a^2}}\hat{\mathbf{x}} \\ &= \frac{1}{4\pi\epsilon_0}\int \frac{x\,dq}{(x^2+a^2)^{\frac{3}{2}}}\hat{\mathbf{x}}\end{aligned}$$

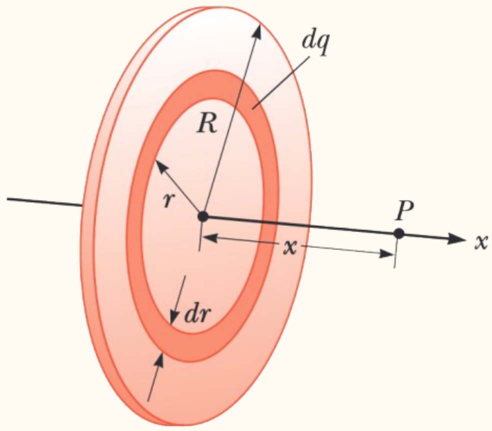
Here, we realise that x and a are constants provided in the question. Hence, we can easily take them out of the integral.

$$\vec{\mathbf{E}} = \frac{1}{4\pi\epsilon_0}\frac{x}{(x^2+a^2)^{\frac{3}{2}}}\int dq\,\hat{\mathbf{x}} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2+a^2)^{\frac{3}{2}}}\,\hat{\mathbf{x}}}$$

Uniformly Charged Disc

Example B1.3

A disc of radius R has a uniform surface charge density σ. Calculate the electric field at a point P that lies along the central perpendicular axis of the disk and a distance x from the center of the disc.



We can approach this problem by considering the disk to be a set of concentric rings. This gives us the agency to use our previous result and sum the contributions of all rings making up the disk.

Each ring element has charge dq, which gives

$$dq = \sigma\,dA = \sigma(2\pi r\,dr)$$

Calculating the electric field contribution,

$$dE_x = \frac{1}{4\pi\epsilon_0}\frac{x}{(x^2+a^2)^{\frac{3}{2}}}(2\pi\sigma r\,dr)$$

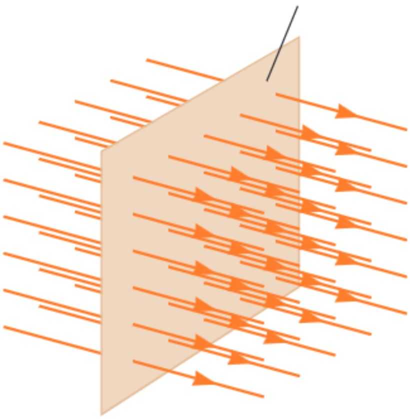
Integrating, we obtain the total electric field.

$$
\begin{aligned}
E_x = \int dE_x &= \int \frac{1}{4\pi\epsilon_0} \frac{x}{(x^2+r^2)^{\frac{3}{2}}} (2\pi\sigma r\, dr) \\
&= \frac{\pi\sigma x}{4\pi\epsilon_0} \int \frac{2r\, dr}{(x^2+r^2)^{\frac{3}{2}}} \\
&= \frac{\pi\sigma x}{4\pi\epsilon_0} \left[\frac{(x^2+r^2)^{-\frac{1}{2}}}{-1/2} \right]_0^R \\
&= \boxed{\frac{2\pi\sigma}{4\pi\epsilon_0} \left(1 - \frac{x}{(R^2+x^2)^{\frac{1}{2}}} \right)}
\end{aligned}
$$

B1.3 Electric Flux

In this section, we treat electric field lines more quantitatively.

Consider an electric field that is uniform in both magnitude and direction. The field lines penetrate a rectangular area of a plane perpendicular to the direction of the electric field, as illustrated below.



The field line density is proportional to the magnitude of the electric field. Hence, the total number of lines penetrating the surface is *proportional* to the product EA.

This product of the magnitude of the electric field and surface area perpendicular to the field is called the **electric flux**, Φ_E, of units $(\mathrm{N\,m^2\,C^{-1}})$.

For the above scenario, the electric flux through $\vec{\mathbf{A}}$ is

$$\Phi_E = EA$$

For a plane tilted at an angle θ to the uniform electric field, the electric flux is then

$$\Phi_E = EA\cos\theta = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}}$$

For more general situations, the electric field may vary over a large surface. This general surface can be divided into a large number of small elements, each of area ΔA_i, with corresponding vector $\Delta\vec{\mathbf{A}}_\mathbf{i}$

whose magnitude represents the area of the ith element of the surface and direction perpendicular to the surface.

This gives us

$$\Phi_{E,i} = \vec{\mathbf{E}}_\mathbf{i} \cdot \Delta\vec{\mathbf{A}}_\mathbf{i}$$

Definition B1.3: Electric Flux

Taking the limit to zero, the sum is replaced by an integral, giving us the **general definition of electric flux**.

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

where $\oint$ denotes a **surface integral**. The value of Φ_E depends on both the field pattern and the surface.

Due to the definition of the area vector, the **net electric flux** through the surface is proportional to the **net number of lines leaving** the surface — this net number refers to the number of lines leaving the surface minus the number of lines entering the surface.

- If more lines are leaving then entering, net electric flux is **positive**.
- If more lines are entering then leaving, net electric flux is **negative**.

These conventions give us

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E_n \, dA$$

where E_n represents the component of the electric field normal to the surface.

B1.4 Gauss's Law

The integrations involved with using Coulomb's law are cumbersome. To avoid this, we can apply Gauss's law, a mathematical alternative to Coulomb's law, to efficiently calculate the electric field of symmetrical objects.

B1.4.1 Explanation

Gauss's law describes a general relationship between the net electric flux through a closed surface (or **gaussian surface**) and the charge enclosed by the surface.

From the derivation of electric flux, we can infer that the flux through any *closed* surface is a measure of the total charge inside it — this is because the field lines that originate on a positive charge must either pass out through the surface, or else terminate on a negative charge inside.

Moreover, any charge *outside* the surface will *not* contribute to the total flux, since its field lines will pass in one side and out the other.

This is the qualitative essence of Gauss's law.

B1.4.2 Derivation

Definition B1.4: Gauss's Law

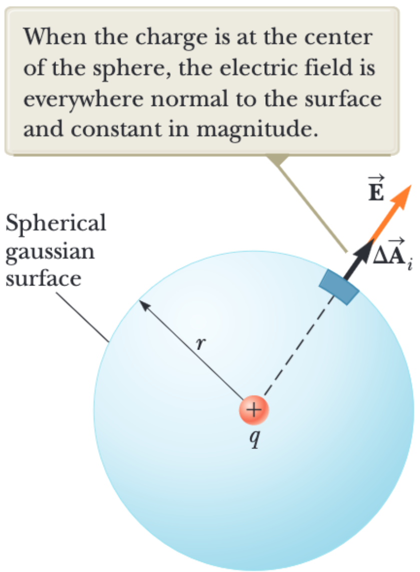
The mathematical form of Gauss's law is

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{enc}}}{\epsilon_0}$$

where $\vec{\mathbf{E}}$ represents the **total** electric field at any point on the surface and q_{enc} represents the net charge enclosed within the surface.

Proof. Consider a sphere enclosing a positive charge $+q$ directly in the centre. The net flux through the spherical gaussian surface is

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E\, dA \cos 0^\circ = E \oint dA$$



E has moved outside of the integral because, by symmetry, E is constant over the surface. Recall that $E = \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}$. Due to the spherical surface, $\oint dA = A = 4\pi r^2$. Hence, the net flux through the spherical gaussian surface is

$$\Phi_E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}(4\pi r^2) = \frac{q}{\epsilon_0}$$

This equation actually holds for *all gaussian surfaces*. The net flux through *any* closed surface surrounding a point charge q is given by q/ϵ_0 and is *independent* of the shape of that surface. $\square$

In practice, Gauss's law can be solved for $\vec{\mathbf{E}}$ in highly symmetrical situations, such as charge distributions that possess spherical, cylindrical, or planar symmetry.

B1.4.3 Applications

The following examples demonstrate ways of choosing the gaussian surface over which the surface integral can be simplified.

In choosing the surface, take advantage of the symmetry of the charge distribution so that E can be removed from the integral. To do so, determine a surface that satisfies at least one of the following conditions.

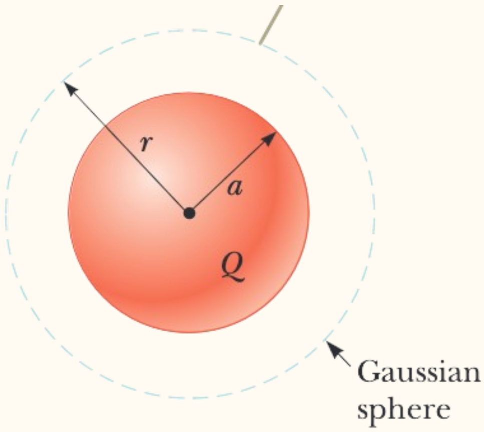
1. The value of the electric field is constant over the surface due to symmetry.
2. $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ are parallel, hence allowing $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E\,dA$.
3. $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ are perpendicular, hence allowing $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 0$.
4. The electric field is zero over the surface.

Spherically Symmetric Charge Distributions

Example B1.4

Consider a solid sphere of radius a with uniform charge density ρ carrying a total positive charge Q. Calculate the magnitude of the electric field at a point P for cases outside and inside the sphere.

For *points outside the sphere*, we can draw another gaussian surface, with r being the distance from the enclosed charge to the point P.



Because the charge is distributed uniformly throughout the sphere, the charge distribution has spherical symmetry and we can apply Gauss's law.

Replacing $\vec{\mathbf{E}} \cdot d\mathbf{A}$ in Gauss's law with $E\,dA$,

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E\,dA = \frac{Q}{\epsilon_0}$$

By symmetry, E has the same value everywhere.

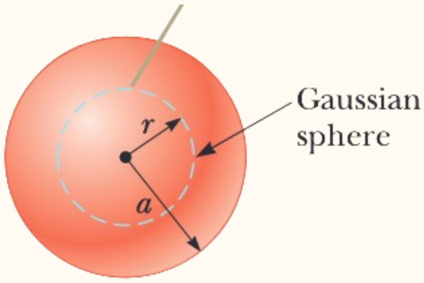
$$\Phi_E = \oint E\,dA = E \oint dA = E(4\pi r^2) = \frac{q_{\text{enc}}}{\epsilon_0}$$

Hence, we can solve for E.

$$\therefore \boxed{E = \frac{q_{\text{enc}}}{4\pi\epsilon_0 r^2} \text{ (where } r > a)}$$

This field is identical to that for a point charge. Therefore, the electric field due to a uniformly charged sphere in the region external to the sphere is *equivalent* to that of a point charge located at the centre of the sphere.

This calculation differs for *points inside the sphere.*



For a spherical gaussian surface having a radius $r < a$ that encloses a volume of V', the enclosed charge q_{enc} is smaller than the total charge Q.

$$q_{\text{enc}} = \rho V' = \rho\left(\frac{4}{3}\pi r^3\right)$$

Since the surface is still spherically symmetrical,

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E\oint dA = E(4\pi r^2) = \frac{q_{\text{enc}}}{\epsilon_0}$$

Solving for E and substituting q_{enc},

$$E = \frac{q_{\text{enc}}}{4\pi\epsilon_0 r^2} = \frac{\rho(\frac{4}{3}\pi r^3)}{4\pi\epsilon_0 r^2} = \frac{\rho}{3\epsilon_0}r$$

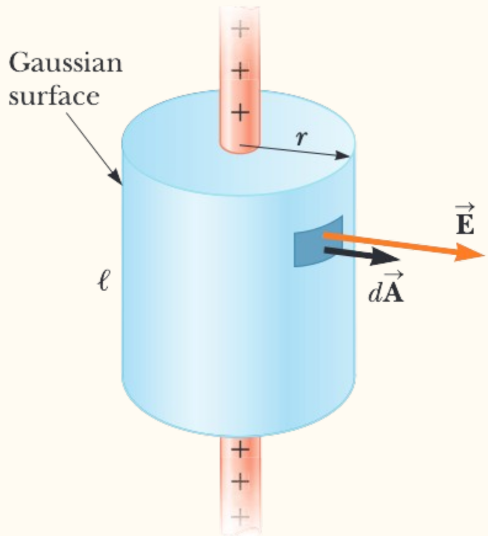
Since the total charge $Q = \rho \times \frac{4}{3}\pi a^3$,

$$\therefore E = \frac{Q/\frac{4}{3}\pi a^3}{3\epsilon_0}r = \boxed{\frac{Q}{4\pi\epsilon_0 a^3}r \text{ (where } r < a)}$$

Cylindrically Symmetric Charge Distribution

Example B1.5

Find the electric field a distance r from a line of positive charge of infinite length and constant charge per unit length λ.



Since the line of charge is infinitely long, the field is the same at all points equidistant from the line.

To reflect the symmetry of the charge distribution, we choose a cylindrical gaussian surface of radius r and length l. For the curved part, $\vec{\mathbf{E}}$ is constant and perpendicular to the surface.

For the flat ends of the cylinder, $\vec{\mathbf{E}}$ is *perpendicular* to $d\vec{\mathbf{A}}$, hence $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 0$ at those surfaces. This allows us to *focus on the curved surfaces only.*

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E \oint dA = EA = \frac{q_{\text{enc}}}{\epsilon_0} = \frac{\lambda l}{\epsilon_0}$$

Substituting the area $A = 2\pi r l$,

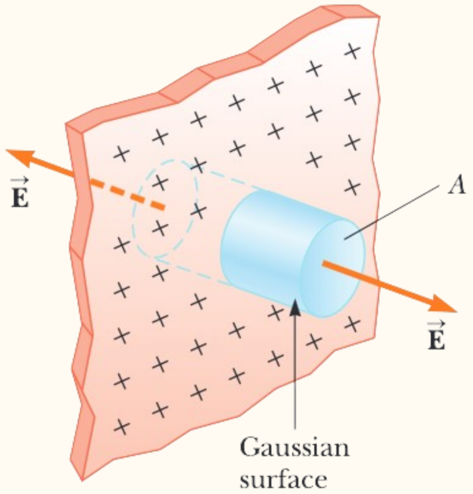
$$E(2\pi r l) = \frac{\lambda l}{\epsilon_0} \implies \boxed{E = \frac{\lambda}{2\pi\epsilon_0 r}}$$

Planarly Symmetric Charge Distribution

Example B1.6

Find the electric field due to an infinite plane of positive charge with uniform surface charge density σ.

By symmetry, $\vec{\mathbf{E}}$ must be perpendicular to the plane at all points. Here, we can characterise a cylindrically-shaped gaussian surface (or gaussian "pillbox") whose axis is perpendicular to the plane and whose ends each have an area A and are equidistant from the plane.



$\vec{\mathbf{E}}$ is parallel to the curved surface and thus causes no electric flux on the curved parts.

For the flat ends of the cylinder, $\vec{\mathbf{E}}$ is constant and parallel to $d\vec{\mathbf{A}}$.

The flux through each end of the cylinder is EA, hence the total flux through the entire gaussian surface is double that, $\Phi_E = 2EA$.

$$\Phi_E = 2EA = \frac{q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

Solving for E, we obtain

$$\therefore \boxed{E = \frac{\sigma}{2\epsilon_0}}$$

B1.5 Conductors in Electrostatic Equilibrium

A good electrical conductor contains charges that are not bound to any atom and therefore are free to move about within the material.

When there is no net motion of charge within a conductor, the conductor is in **electrostatic equilibrium**. Such conductors have the following properties.

1. The $\vec{\mathbf{E}}$ field is *zero everywhere* inside the conductor, whether the conductor is solid or hollow.
2. If the conductor is isolated and carries a charge, the charge resides on its surface.
3. The electric field at a point just outside a charged conductor is perpendicular to the surface of the conductor and has a magnitude of σ/ϵ_0, where σ is the surface charge density at that point.
4. The surface of a conductor is an emphequipotential surface, and the electric potential is *constant everywhere* inside the conductor and equal ot its value at the surface.
5. On an irregularly-shaped conductor, the surface charge density is greatest at locations where the radius of curvature of the surface is smallest.

B1.6 Electric Potential

When a charge q is placed in an electric field $\vec{\mathbf{E}}$, the particle experiences an electric force $\vec{\mathbf{F}} = q\vec{\mathbf{E}}$. This force is *conservative* because the force between charges described by Coulomb's law is conservative.

For an infinitesimal displacement $d\vec{\mathbf{s}}$ of a point charge q in an electric field, the work done within the charge-field system by the electric field on the charge is $W_{\text{int}} = \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = q\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$.

As learnt previously, the internal work done in a system is equal to the negative of the change in the potential energy of the system, or $W_{\text{int}} = -\Delta U$.

Therefore, as the charge q is displaced, the electric potential energy of the charge-field system is changed by an amount $dU = -W_{\text{int}} = -q\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$.

Definition B1.5: Electric Potential Energy (H3)

For a finite displacement of the charge from some point A in space to some other point B, the *change in electric potential energy of the system* is

$$\Delta U = -q \int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r}$$

The integration is performed along the path that q follows as it moves from A to B. As the electric force is conservative, the line integral does not depend on the specific path taken from A to B.

Definition B1.6: Electric Potential (H3)

Dividing the potential energy by the charge gives a physical quantity that depends only on the source charge distribution and has a value at every point at an electric field. This quantity is called the **electric potential**.

$$V = \frac{U}{q}$$

Both potential energy and potential are scalar quantities.

Definition B1.7: Potential Difference (H3)

The **potential difference** $\Delta V = V_B - V_A$ between two points A and B in an electric field is defined as the charge in electric potential energy of the system when a charge q is moved between the points, divided by the charge.

$$\Delta V = \frac{\Delta U}{q} = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

The potential difference is *not equivalent* to the difference in potential energy.

Potential difference between A and B exists solely due to a source charge — potential energy requires at least two charges.

Now, consider the situation where an *external agent* moves a charge from A to B with changing the kinetic energy.

This agent performs work that changes the potential energy, where, by the work-energy theorem, $W = \Delta U$.

Definition B1.8: Work Done for Charges

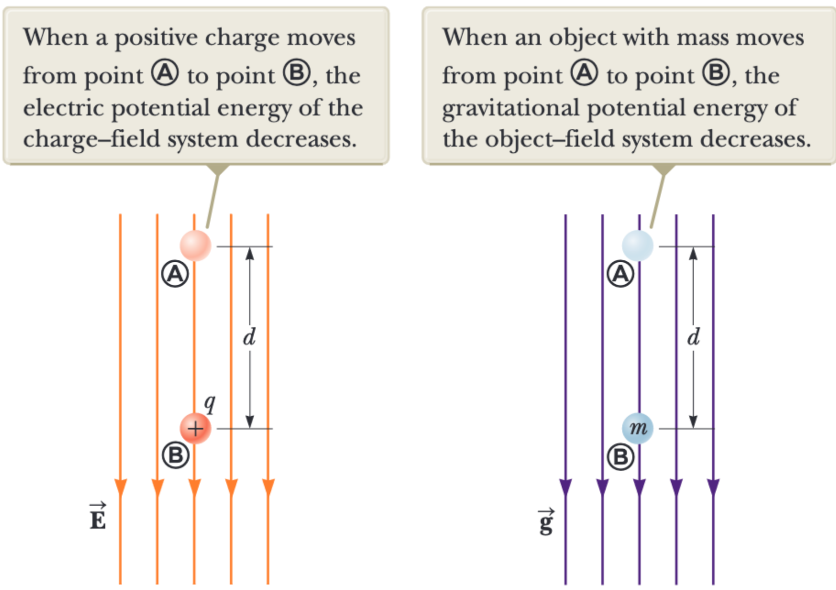
The **work done** by an external agent in moving a charge q through an electric field at constant velocity is thus

$$W = q\Delta V$$

B1.6.1 Potential in Uniform Electric Field

From the previous equation, we can highlight an important point regarding units.

- Electrical potential and potential difference have S.I. unit volt (V) or joules per coulomb.
- However, as defined earlier, potential difference also has units of *electric field multiplied by distance*.
- Thus, the S.I. unit of electric field can be expressed in volts per meter $1\ \mathrm{N\,C^{-1}} = 1\ \mathrm{V\,m^{-1}}$.
- This shows that the electric field is a measure of the *rate of change of the electrical potential* with respect to position.



We can calculate the potential difference between points A and B separated by a distance d, where the displacement $\vec{\mathbf{s}}$ points from A to B and is parallel to the field lines.

$$\Delta V = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\int_A^b E\,ds(\cos 0) = -\int_A^B E\,ds$$

Since E is constant, it can be removed from the integral sign, giving

$$\Delta V = -E\int_A^B ds = -Ed$$

This shows that electric field lines *always point in the direction of decreasing electric potential.*

Suppose a charge q moves from A to B. The change in potential energy is given by

$$\Delta U = q\Delta V = -qEd$$

This shows if q is positive, then ΔU is negative, and vice versa. In a system consisting of a positive charge and an electric field, the electric potential energy of the system *decreases* when the charge moves in the direction of the field.

B1.6.2 Potential for Continuous Charges

The electric potential due to a *single point charge* q at any distance r from the charge is

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r} \implies U = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}$$

For *multiple discrete point charges*,

$$V = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}, \quad U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}$$

For a *continuous charge distribution*,

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r}$$

and the total potential energy $\sum U$ is calculated by integrating dU for *every pair of charges.*

Note that the electric field can derived by taking the negative derivative of the electric potential.

$$E_r = -\frac{dV}{dr}$$

B1.7 Electric Dipole

Definition B1.9: Electric Dipole

An **electric dipole** consists of two charges of equal magnitude and opposite sign separated by a certain distance. It is common to denote this distance as $2a$.

Definition B1.10: Electric Dipole Moment

The **electric dipole moment** of an electric dipole configuration is defined as the vector $\vec{\mathbf{p}}$ directed from the negative charge $-q$ to the positive charge $+q$ along the line joining the charges, which is defined to be

$$\vec{\mathbf{p}} \equiv 2\vec{\mathbf{a}}q$$

where $2\vec{\mathbf{a}}$ is the separation between the two electric charges.

Definition B1.11: Torque of Electric Dipole

For an electric dipole placed in an electric field $\vec{\mathbf{E}}$ at an oblique angle θ with the field, the **net torque** created about the midpoint **of the dipole** is given as

$$\tau = \vec{\mathbf{p}} \times \vec{\mathbf{E}} = pE \sin\theta$$

Proof. Suppose an electric dipole with separation $2a$ is placed in a uniform electric field $\vec{\mathbf{E}}$, whilst making an angle θ with the field. The $\vec{\mathbf{E}}$ field is *external* to the dipole, and not the field that is created by the dipole.

Within this electric field, each charge experiences an equal and opposite force pointing in the direction of the external electric field. Though the net force acting on the dipole is zero, these two forces produce a net torque about the midpoint of the two charges.

The torque due to the positive charge has magnitude $Fa\sin\theta$. The torque due to the negative charge has magnitude $Fa\sin\theta$ as well. Therefore, the magnitude of the net torque about its midpoint is

$$\begin{aligned} \tau &= 2Fa\sin\theta = 2aqE\sin\theta \\ &= pE\sin\theta \end{aligned}$$

□

B1.7.1 Potential Energy of Electric Dipoles

Definition B1.12: Potential Energy of Electric Dipole

For an electric dipole placed in an electric field $\vec{\mathbf{E}}$ with electric dipole moment $\vec{\mathbf{p}}$, the **potential energy possessed by said dipole** is given as

$$U_E = -\vec{\mathbf{p}} \cdot \vec{\mathbf{E}}$$

The potential energy of any electric dipole is associated with a *rotational configuration* of the system. Harkening back to our understanding of rotational mechanics, the work dW required to rotate the dipole through an angle $d\theta$ is $dW = \tau d\theta$.

Hence, for a rotation from θ_i to θ_f, the change in potential energy of the system is

$$\begin{aligned} \Delta U = U_f - U_i = \int_{\theta_i}^{\theta_f} \tau\, d\theta &= \int_{\theta_i}^{\theta_f} pE\sin\theta\, d\theta \\ &= pE\int_{\theta_i}^{\theta_f} \sin\theta\, d\theta \\ &= pE[-\cos\theta]_{\theta_i}^{\theta_f} \\ &= pE(\cos\theta_i - \cos\theta_f) \end{aligned}$$

Choosing $\theta_i = \pi/2$ and $\theta_f = \theta$, this expression can be further simplified into

$$\Delta U = -pE\cos\theta$$

B1.8 Magnetic Fields

Definition B1.13: Gauss's Law in Magnetism

Gauss's law in magnetism states that the net magnetic flux through any closed surface is always zero.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

This implies that magnetic monopoles do not exist.

B1.8.1 Ampere's Law

Definition B1.14: Ampere's Law

Ampere's law states the the line integral of $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ around any closed path is equal to $\mu_0 I$, where I is the total current passing through any surface bounded by the closed linear path.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I$$

Long Wire

Example B1.7

A long straight wire of radius R carries a steady current I that is uniformly distributed through the cross-section of the wire. Calculate the magnetic field at a distance r from the centre of the wire in the regions inside the wire and outside the wire.

When **inside** the wire, $r < R$, the current I' within the closed loop 2 is such that

$$\frac{I'}{I} = \frac{J \times \pi r^2}{J \times \pi R^2} = \frac{r^2}{R^2} \quad \left(J = \frac{I}{A}\right)$$

Therefore, by Ampere's law,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B \oint ds = B(2\pi r) = \mu_0 I' = \mu_0 \left(\frac{r^2}{R^2} I\right)$$

$$\implies \boxed{B = \frac{\mu_0 I}{2\pi R^2} r}$$

When **outside** the wire, $r \geq R$.

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B \oint ds = B(2\pi r) = \mu_0 I$$

$$\implies \boxed{B = \frac{\mu_0 I}{2\pi r}}$$

Toroid

Example B1.8

A toroid, consisting of wire wrapped around a non-conducting ring, is often used to create a near-uniform magnetic field in some enclosed area. For a toroid having N closely-spaced turns of wire, determine the magnetic field in the region occupied by the torus, a distance r from the centre.

Consider the circular loop 1 of radius r within the ring. By symmetry, the magnitude of the field B is constant in this circle and tangential to it ($\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B\,ds$). Furthermore, the total current is NI.

Applying Ampere's law,

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = B \oint ds = B(2\pi r) = \mu_0 NI$$
$$\implies \boxed{B = \frac{\mu_0 NI}{2\pi r}}$$

Solenoid

Example B1.9

A solenoid is a long wire wound in the form of a helix. Considering an ideal solenoid with N turns, find the magnitude of the uniform magnetic field created within the interior.

Consider loop 2 of length ℓ and width w. Applying Ampere's law, we obtain

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I \implies B\ell = \mu_0 NI$$
$$\implies \boxed{B = \mu_0 \frac{N}{l} I}$$

B1.9 Magnetic Dipoles

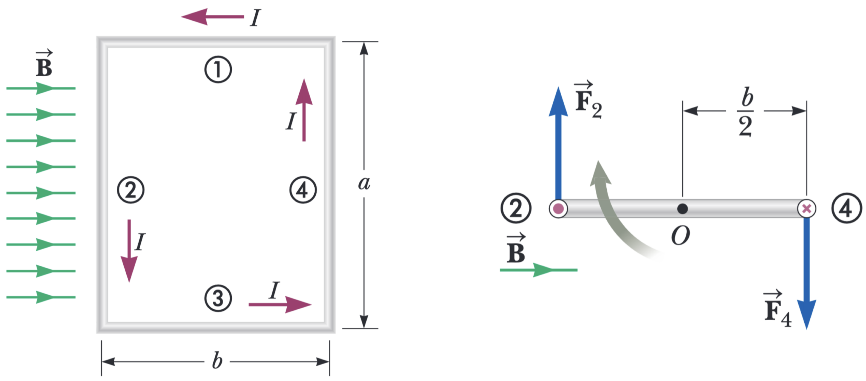
We begin our analysis of magnetic forces acting magnetic dipoles by first considering a current-carrying conductor.

Take a rectangular loop carrying a current I in the presence of a uniform magnetic field directed *parallel to the plane* of the loop.

Magnetic forces only act on the sides that are oriented perpendicular to the field (that is, the sides of length a). The magnitudes of said forces are

$$F_2 = F_4 = BI\ell = BIa$$

We also notice that these forces impose a net torque on the rectangular loop, rotating it about the



centre O. The magnitude of this torque is given by

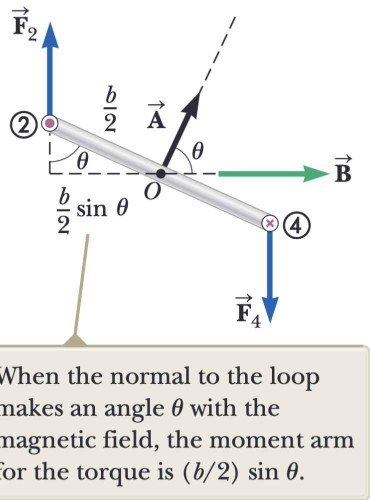
$$\begin{aligned} \tau = F_2 \cdot \frac{b}{2} + F_4 \cdot \frac{b}{2} &= BIa \cdot \frac{b}{2} + BIa + \frac{b}{2} \\ &= IabB \\ &= IAB \end{aligned}$$

where A is the area enclosed by the loop.

Where the loop is oriented at an oblique angle θ with respect to the direction of the magnetic field, the magnitude of the resultant torque about O is given by

$$\tau = IAB\sin\theta$$

as seen from the diagram below, where the perpendicular between O and the line of the magnetic force is $b/2 \cdot \sin\theta$ for each side of the loop.



Definition B1.15: Magnetic Dipole Moment

The product $I\vec{\mathbf{A}}$ is defined to the **magnetic dipole moment** (also **magnetic moment**) of the loop.

$$\vec{\boldsymbol{\mu}} = I\vec{\mathbf{A}}$$

where $\vec{\mathbf{A}}$ is the *area vector* of the loop.

For a coil with N loops of the same area, the magnetic moment of the coil is then

$$\vec{\boldsymbol{\mu}} = NI\vec{\mathbf{A}}$$

The S.I. unit of magnetic dipole moment is the ampere square meter, $\mathrm{A\,m^{-2}}$.

Definition B1.16: Torque on Current-Carrying Loop

To simplify our derivation of torque even further, the **torque exerted on a current-carrying loop** placed in a uniform magnetic field $\vec{\mathbf{B}}$ is given by

$$\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}$$

where $\vec{\boldsymbol{\mu}}$ is the magnetic moment of the loop.

Definition B1.17: Potential Energy of Magnetic Dipole

In a similar vein with electric dipoles, the **potential energy of a magnetic dipole** in a magnetic field is given by

$$U_B = -\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}}$$

which has the lowest energy when $\vec{\boldsymbol{\mu}}$ is *parallel* to $\vec{\mathbf{B}}$, and has the highest energy when $\vec{\boldsymbol{\mu}}$ is *anti-parallel* to $\vec{\mathbf{B}}$.